ON A HEATH-JARROW-MORTON APPROACH FOR STOCK OPTIONS

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ABSTRACT. This paper aims at transferring the philosophy behind Heath-Jarrow-Morton to the modelling of call options with all strikes and maturities. Contrary to the approach by Carmona and Nadtochiy [8] and related to the recent contribution [10] by the same authors, the key parametrisation of our approach involves time-inhomogeneous Lévy processes instead of local volatility models. We provide necessary and sufficient conditions for absence of arbitrage. Moreover we discuss the construction of arbitrage-free models. Specifically, we prove their existence and uniqueness given basic building blocks.

KEYWORDS: Heath-Jarrow-Morton, option price surfaces, Lévy processes

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1. INTRODUCTION

The traditional approach to modelling stock options takes the underlying as a starting point. If the dynamics of the stock are specified under a risk neutral measure for the whole market (i.e. all discounted asset price processes are martingales), then options prices are obtained as conditional expectations of their payoff. In reality, standard options as calls and puts are liquidly traded. If one wants to obtain vanilla option prices which are consistent with observed market values, special care has to be taken. A common and also theoretically reasonable way is calibration, i.e. to choose the parameters for the stock dynamics such that the model approximates market values sufficiently well. After a while, models typically have to be recalibrated, i.e. different parameters have to be chosen in order for model prices to be still consistent with market quotes. However, frequent recalibration is unsatisfactory from a theoretical point of view because model parameters are meant to be deterministic and constant. Its necessity indicates that the chosen class fails to describe the market consistently.

In Markovian factor models with additional unobservable state variables, the situation is slightly more involved. Since these state variables are randomly changing within the model, they may be recalibrated, which means that their current values are inferred from option prices. In practice, however, the model parameters are often recalibrated as well because the few state variables do not provide enough flexibility to match observed option data. In this case, we are facing the same theoretically unsatisfactory situation as above.

A possible way out is to model the whole surface of call options as a state variable, i.e. as a family of primary assets in their own right. This alternative perspective is motivated from the Heath-Jarrow-Morton (HJM, see [22]) approach in interest rate theory. Rather than considering bonds as derivatives on the short rate, HJM treat the whole family of zero bonds or equivalently the forward rate curve as state variable in the first place. In the context of HJM-type approaches for stock options, Wissel [37] and Schönbucher [35] consider the case of a single strike, whereas Cont et al. [13] and Carmona and Nadtochiy [8, 10] allow for all strikes and maturities. Further important references in this context include Jacod and Protter [24], Schweizer and Wissel [36] and Wissel [39]. The HJM approach has been adapted to other asset classes, e.g. credit models in Benanni [5], Schönbucher [35], or Sidenius et al. [38] and variance swaps in Bühler [6], cf. Carmona [7] for an overview and further references.

Similar to Carmona and Nadtochiy [8] we aim at modelling the whole call option price surface using the HJM methodology. However, our approach differs in the choice of the parametrisation or *codebook*, which constitutes a crucial step in HJM-type setups. By relying on time-inhomogeneous Lévy processes rather than Dupire's local volatility models, we can avoid some intrinsic difficulties of the framework in [8]. E.g., a simpler drift condition makes the approach more amenable to existence and uniqueness results. Moreover, the Lévy-based setup allows for jumps and may hence be more suitable to account particularly for short-term option prices, cf. [14, Section 1.2].

More recently and independently of the present study, Carmona and Nadtochiy [10] have also put forward a HJM-type approach for the option price surface which is based on time-inhomogeneous Lévy processes. The similarities and differences of their and our approach are discussed in Section 5.

The paper is arranged as follows. We start in Section 2 with an informal discussion of the HJM philosophy, as a motivation to its application to stock options. Section 3 provides necessary and sufficient conditions for an option surface model to be arbitrage-free or, more precisely, risk-neutral. Subsequently, we turn to existence and uniqueness of option surface models given basic building blocks. In particular, we provide a concrete example which turns out to be related to the stochastic volatility model proposed by Barndorff-Nielsen and Shephard in [2]. Mathematical tools and some technical proofs are relegated to the appendix. Facts on seminartingale characteristics are summarised in Section A. The subsequent section concerns mainly option pricing by Fourier transform. In Section C we consider stochastic differential equations in Fréchet spaces, driven by subordinators. This framework is needed for existence and uniqueness results in Section 4.

Notation. Re and Im denote the real resp. imaginary part of a complex vector in $\mathbb{C}^d$. We write $[a,b]$ for the closed interval $\{x \in \mathbb{R} : a \leq x \leq b\}$, which is empty if $a > b$. We use the notations ∂_u and D for partial and total derivatives, respectively. We often write $\beta \bullet X_t = \int_0^t \beta_s dX_s$ for stochastic integrals. $L(X)$ denotes the set of X-integrable predictable processes for a semimartingale X. If we talk about an $m+n$-dimensional semimartingale (X,Y), we mean that X is an $\mathbb{R}^m$-valued semimartingale and Y is an $\mathbb{R}^n$-valued semimartingale. For $u, v \in \mathbb{C}^d$ we denote the bilinear form of u and v by $uv := \sum_{k=1}^d u_k v_k$. The abbreviation *PII* stands for processes with independent increments in the sense of [25]. Further unexplained notation is used as in [25].

2. HEATH-JARROW-MORTON AND LÉVY MODELS

This section provides an informal discussion of the HJM philosophy and its application to stock options.

2.1. **The Heath-Jarrow-Morton philosophy.** According to the fundamental theorem of asset pricing, there exists at least one equivalent probability measure that turns discounted prices of all traded securities into martingales or, more precisely, into σ-martingales. For simplicity we take the point of view of risk-neutral modelling in this paper, i.e. we specify the dynamics of all assets in the market directly under such an equivalent martingale measure (EMM). Moreover, we assume the existence of a deterministic bank account unless we refer to the original HJM setup in interest rate theory. This allows us to express all prices easily in discounted terms.

Before we turn to our concrete setup, we want to highlight key features of the HJM approach in general. For more background and examples we refer the reader to the brilliant exposition from Carmona [7], which greatly inspired our work. We proceed by stating seven informal axioms or steps.

(1) In HJM-type setups there typically exists a canonical underlying asset or reference process, namely the money market account in interest rate theory or the stock in the present paper. The object of interest, namely bonds in interest rate theory or vanilla options in stock markets, can be interpreted as derivatives on the canonical process. HJM-type approaches typically focus on a whole manifold of such — at least in theory — liquidly traded derivatives, e.g. the one-parameter manifold of bonds with all maturities or the two-parameter manifold of call options with all strikes and maturities. As first and probably most important HJM axiom we claim that this manifold of liquid derivatives is to be treated as the set of primary assets. It — rather than the canonical reference asset — constitutes the object whose dynamics should be modelled in the first place.

Example 1. Zero bonds are securities with terminal value 1 and appear to be somewhat degenerate derivatives. But as noted above, we consider discounted prices in this paper. If the money market account S^0 is chosen as numeraire, the discounted payoff of a bond maturing at T is of the form $1/S^0_T$ and hence a function of S^0. In this broader sense, we view bonds here as derivatives on the money market account.

Example 2. European call options on a stock S with maturity T and strike K have a payoff of the form $(S_T - K)^+$. The same is true for their discounted payoff relative to a deterministic numeraire, provided that S and K are replaced by their discounted analogues as well.

(2) The first axiom immediately leads to the second one: do *not* model the canonical reference asset in detail under the market's risk-neutral measure. Indeed, otherwise all derivative prices would be entirely determined by their martingale property, leaving no room for a specification of their dynamics.

(3) Direct modelling of the above manifold typically leads to awkward constraints. Zero bond price processes must terminate in 1, vanilla options in their respective payoff. Rather than prices themselves one should therefore consider a convenient parametrisation (or *codebook* in the language of Carmona [7]), e.g. instantaneous forward rates in interest rate theory. Specifying the dynamics of this codebook leads immediately to a model for the manifold of primary assets. If the codebook is properly chosen, then static arbitrage constraints are satisfied automatically, cf. Steps 4 and 5.

(4) It is generally understood that choosing a convenient parametrisation constitutes a crucial step for a successful HJM-type approach. This is particularly obvious in the context of call options. Their prices are linked by a number of non-trivial static arbitrage constraints, which must hold independently of any particular model, cf. Davis and Hobson [15]. These static constraints have to be respected by any codebook dynamics. Specifying the latter properly may therefore be a difficult task unless the codebook is chosen such that the constraints naturally hold. We now suggest a way how to come up with a reasonable parametrisation.

The starting point is a family of simple risk-neutral models for the canonical underlying whose parameter space has — loosely speaking — the same "dimension" or "size" as the space of liquid derivative manifolds. Provided sufficient regularity holds, the presently observed manifold of derivative prices is explained by one and only one of these models.

Example 1. In interest rate theory consider bank accounts of the form

$$S_t^0 = \exp\left(\int_0^t r(s)ds\right) \tag{2.1}$$

with deterministic short rate $r(T), T \in \mathbb{R}_+$. Fix $t \in \mathbb{R}_+$ and a differentiable curve of bond prices $B(t,T)$, $T \in \mathbb{R}_+$. Except for the past up to time t, the observed bond prices are consistent with one and only one of these models, namely for

$$r(T) := -\partial_T \log(B(t,T)). \tag{2.2}$$

Example 2. Consider Dupire's local volatility models

$$dS_t = S_t \sigma(S_t, t) dW_t$$

for a discounted stock, where W denotes standard Brownian motion and $\sigma : \mathbb{R}_+^2 \to \mathbb{R}$ a deterministic function. Up to regularity, any surface of discounted call option prices $C_t(T,K)$ with varying maturity T and strike K and fixed current date $t \in \mathbb{R}_+$ is obtained by one and only one local volatility function σ, namely by

$$\sigma^2(K,T) := \frac{2\partial_T C_t(T,K)}{K^2 \partial_{KK} C_t(T,K)}. \tag{2.3}$$

Note that the above starting point should not be taken as a precise mathematical requirement. As illustrated in the examples, we relate "size" liberally to the number of arguments in the respective functions: parameter curves correspond to bond price curves, parameter surfaces to option price surfaces. The actual regularity needed for one-to-one correspondence crucially depends on the chosen family of simple models.

If market data follows the simple model, the parameter manifold, e.g. $(r(T))_{T \geq t}$ in Example 1, is deterministic and does not depend on the time t when derivative prices are observed. Generally, however, market data does not follow such a simple model as in the two examples. Hence, evaluation of the right-hand side of (2.2) and (2.3) leads to a parameter manifold which changes randomly over time.

Example 1. The *instantaneous forward rate* curve

$$f(t,T) := -\partial_T \log(B(t,T)), \quad T \geq t$$

for fixed $t \in \mathbb{R}_+$ can be interpreted as the family of deterministic short rates that is consistent with the presently observed bond price curve $B(t,T)$, $T \geq t$.

Example 2. The *implied local volatility*

$$\sigma^2(K,T) := \frac{2\partial_T C_t(T,K)}{K^2 \partial_{KK} C_t(T,K)}, \quad K > 0, \quad T \geq t$$

for fixed $t \in \mathbb{R}_+$ can be interpreted as the unique local volatility function that is consistent with the presently observed discounted call prices $C_t(T,K)$, $T \geq t$, $K > 0$.

The idea now is to take this present parameter manifold as a parametrisation or codebook for the manifold of derivatives.

(5) In a next step, we set this parameter manifold "in motion." We consider the codebook, e.g. the instantaneous forward rate curve $f(t,T)$ or the implied local volatility

$\sigma_t(T,K)$, as an infinite-dimensional stochastic process. It is typically modelled by a stochastic differential equation, e.g.

$$df(t,T) = \alpha(t,T)dt + \beta(t,T)dW_t,$$

where W denotes standard Brownian motion. As long as the solution to this equation moves within the parameter space for the family of simple models, one automatically obtains derivative prices that satisfy any static arbitrage constraints. Indeed, since the current bond prices resp. call prices coincide with the prices from an arbitrage-free model, they cannot violate any such constraints, however complicated they might be. This automatic absence of static arbitrage motivates the codebook choice in Step 4.

(6) Absence of static arbitrage does not imply absence of arbitrage altogether. Under the risk-neutral modelling paradigm, all discounted assets must be martingales. In interest rate theory this leads to the well known HJM drift condition. More generally it means that the drift part of the codebook dynamics of Step 5 is determined by its diffusive component.

(7) Finally we come back to Step 2. The dynamics of the canonical reference asset process is typically implied by the current state of the codebook. E.g. in interest rate theory the short rate is determined by the so-called consistency condition

$$r(t) = f(t,t).$$

Similar conditions determine the current stock volatility in [37, 39, 8, 10].

2.2. **Time-inhomogeneous Lévy models.** According to the above interpretation, the approach of [8] to option surface modelling relies on the family of Dupire's local volatility models. Similarly as the independent study [10], we suggest another family of simple models for the stock, also relying on a two-parameter manifold. To this end, suppose that the discounted stock is a martingale of the form $S = e^X$, where the *return process* X denotes a process with independent increments (or time-inhomogeneous Lévy process, henceforth PII) on some filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$. Recall that we work with risk-neutral probabilities, i.e. discounted asset prices are supposed to be P-martingales. More specifically, the characteristic function of X is assumed to be absolutely continuous in time, i.e.

$$E(e^{iuX_t}) = \exp\left(iuX_0 + \int_0^t \Psi(s,u)ds\right) \tag{2.4}$$

with some function $\Psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$.

We assume call options of all strikes and maturities to be liquidly traded. Specifically, we write $C_t(T,K)$ for the discounted price at time t of a call which expires at T with discounted strike K. A slight extension of [4, Proposition 1] shows that option prices can be expressed in terms of Ψ. To this end, we define *modified option prices*

$$\mathscr{O}_t(T,x) := e^{-(x+X_t)}C_t(T, e^{x+X_t}) - (e^{-x}-1)^+.$$

Since call option prices are obtained from $C_t(T,K) = E((S_T-K)^+|\mathscr{F}_t)$, by call-put parity, and by $E(S_T|\mathscr{F}_t) = S_t$, we have

$$\mathscr{O}_t(T,x) = \begin{cases} E((e^{(X_T-X_t)-x}-1)^+|\mathscr{F}_t) & \text{if } x \geq 0, \\ E((1-e^{(X_T-X_t)-x})^+|\mathscr{F}_t) & \text{if } x < 0. \end{cases}$$

Proposition B.4 yields

$$\mathscr{O}_t(T,x) = \mathscr{F}^{-1}\left\{u \mapsto \frac{1-E(e^{iu(X_T-X_t)}|\mathscr{F}_t)}{u^2+iu}\right\}(x), \tag{2.5}$$

$$\mathscr{F}\{x \mapsto \mathscr{O}_t(T,x)\}(u) = \frac{1-E(e^{iu(X_T-X_t)}|\mathscr{F}_t)}{u^2+iu} \tag{2.6}$$

where $\mathscr{F}^{-1}$ and $\mathscr{F}$ denote the improper inverse Fourier transform and the improper Fourier transform, respectively, in the sense of (B.1, B.2) in Section B.1 of the appendix. Since

$$C_t(T,K) = (S_t-K)^+ + K\mathscr{O}_t\left(T, \log\frac{K}{S_t}\right) \tag{2.7}$$

and

$$E(e^{iu(X_T-X_t)}|\mathscr{F}_t) = \exp\left(\int_t^T \Psi(s,u)ds\right), \tag{2.8}$$

we can compute option prices according to the following diagram:

$$\Psi \to \exp\left(\int_t^T \Psi(s,\cdot)ds\right) \to \mathscr{O}_t(T,\cdot) \to C_t(T,\cdot).$$

For the last step we also need the present stock price S_t. Under sufficient smoothness we can invert all transformations. Indeed, we have

$$\Psi(T,u) = \partial_T \log\left(1-(u^2+iu)\mathscr{F}\{x \mapsto \mathscr{O}_t(T,x)\}(u)\right). \tag{2.9}$$

Hence we obtain option prices from Ψ and vice versa as long as we know the present stock price.

2.3. **Setting Lévy in motion.** Generally we do not assume that the *return process*

$$X := \log(S) \tag{2.10}$$

follows a time-inhomogeneous Lévy process. Hence the right-hand side of Equation (2.9) will typically change randomly over time. In line with Step 4 above, we define modified option prices

$$\mathscr{O}_t(T,x) := e^{-(x+X_t)}C_t(T,e^{x+X_t}) - (e^{-x}-1)^+ \tag{2.11}$$

as before and

$$\Psi_t(T,u) := \partial_T \log\left(1-(u^2+iu)\mathscr{F}\{x \mapsto \mathscr{O}_t(T,x)\}(u)\right). \tag{2.12}$$

This constitutes our *codebook process* for the surface of discounted option prices. As in Section 2.2 the asset price processes S and $C(T,K)$ can be recovered from X and $\Psi(T,u)$ via

$$\begin{aligned} S &= \exp(X), \\ \mathscr{O}_t(T,x) &= \mathscr{F}^{-1}\left\{u \mapsto \frac{1-\exp(\int_t^T \Psi_t(s,u)ds)}{u^2+iu}\right\}(x), \\ C_t(T,K) &= (S_t-K)^+ + K\mathscr{O}_t\left(T,\log\frac{K}{S_t}\right). \end{aligned}$$

In the remainder of this paper we assume that the infinite-dimensional codebook process satisfies an equation of the form

$$d\Psi_t(T,u) = \alpha_t(T,u)dt + \beta_t(T,u)dM_t, \tag{2.13}$$

driven by some finite-dimensional semimartingale M.

3. MODEL SETUP AND RISK NEUTRALITY

As before we fix a filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in\mathbb{R}_+},P)$ with trivial initial σ-field $\mathscr{F}_0$. In this section we single out conditions such that a given pair (X,Ψ) corresponds via (2.10 – 2.12) to a risk-neutral model for the stock and its call options.

3.1. **Option surface models.** We denote by Π the set of characteristic exponents of Lévy processes L such that $E(e^{L_1})=1$. More precisely, Π contains all functions $\psi:\mathbb{R}\to\mathbb{C}$ of the form

$$\psi(u)=-\frac{u^2+iu}{2}c+\int(e^{iux}-1-iu(e^x-1))K(dx),$$

where $c\in\mathbb{R}_+$ and K denotes a Lévy measure on $\mathbb{R}$ satisfying $\int_{\{x>1\}}e^xK(dx)<\infty$.

Definition 3.1. A quintuple $(X,\Psi_0,\alpha,\beta,M)$ is an *option surface model* if

- (X,M) is a $1+d$-dimensional semimartingale that allows for local characteristics in the sense of Section A.1,
- $\Psi_0:\mathbb{R}_+\times\mathbb{R}\to\mathbb{C}$ is measurable with $\int_0^T|\Psi_0(r,u)|dr<\infty$ for any $T\in\mathbb{R}_+$, $u\in\mathbb{R}$,
- $\alpha(T,u),\beta(T,u)$ are $\mathbb{C}$- resp. $\mathbb{C}^d$-valued processes for any $T\in\mathbb{R}_+$, $u\in\mathbb{R}$,
- $(\omega,t,T,u)\mapsto\alpha_t(T,u)(\omega),\beta_t(T,u)(\omega)$ are $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{B}$-measurable, where $\mathscr{P}$ denotes the predictable σ-field on $\Omega\times\mathbb{R}_+$,
- $\int_0^t\int_0^T|\alpha_s(r,u)|drds<\infty$ for any $t,T\in\mathbb{R}_+$, $u\in\mathbb{R}$,
- $\int_0^T|\beta_t(r,u)|^2dr<\infty$ for any $t,T\in\mathbb{R}_+$, $u\in\mathbb{R}$,
- $((\int_0^T|\beta_t^j(r,u)|^2dr)^{1/2})_{t\in\mathbb{R}_+}\in L(M^j)$ for any fixed $T\in\mathbb{R}_+$, $u\in\mathbb{R}$, $j\in\{1,\dots,d\}$, where the set $L(M^j)$ of M^j-integrable processes is defined as in [25, Definition III.6.17],
- a version of the corresponding *codebook process*

$$\Psi_t(T,u):=\Psi_0(T,u)+\int_0^{t\wedge T}\alpha_s(T,u)ds+\int_0^{t\wedge T}\beta_s(T,u)dM_s \tag{3.1}$$

  has the following properties:
  (1) $(\omega,t,T,u)\mapsto\Psi_t(T,u)(\omega)$ is $\mathscr{O}\otimes\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{B}$-measurable, where $\mathscr{O}$ denotes the optional σ-field on $\Omega\times\mathbb{R}_+$,
  (2) $u\mapsto\int_t^T\Psi_s(r,u)(\omega)dr$ is in Π for any $T\in\mathbb{R}_+$, $t\in[0,T]$, $s\in[0,t]$, $\omega\in\Omega$.

Remark 3.2. The square-integrability conditions on β are imposed only to warrant

$$\int_0^T\left|\int_0^t\beta_s(r,u)dM_s\right|dr\ <\ \infty\quad\text{and} \tag{3.2}$$

$$\int_0^t\int_0^T\beta_s(r,u)drdM_s\ =\ \int_0^T\int_0^t\beta_s(r,u)dM_sdr. \tag{3.3}$$

If M has increasing components, we can and do replace the integrability conditions on β by the weaker requirement

- $\sum_{j=1}^d\int_0^t\int_0^T|\beta_s^j(r,u)|drdM_s^j<\infty$ for any $t,T\in\mathbb{R}_+$, $u\in\mathbb{R}$,

which implies (3.2, 3.3) by Fubini's theorem.

We denote the local exponents of $(X,M),X$ by $\psi^{(X,M)},\psi^X$ and their domains by $\mathscr{U}^{(X,M)},\mathscr{U}^X$, cf. Definitions A.4 and A.6. In line with Section 2.3, the discounted stock and call price processes associated with an option surface model are defined by

$$S_t := \exp(X_t), \tag{3.4}$$

$$\mathscr{O}_t(T,x) := \mathscr{F}^{-1}\left\{u \mapsto \frac{1-\exp\left(\int_t^T \Psi_t(r,u)dr\right)}{u^2+iu}\right\}(x), \tag{3.5}$$

$$C_t(T,K) := (S_t-K)^+ + K\mathscr{O}_t\left(T,\log\frac{K}{S_t}\right) \tag{3.6}$$

for any $T\in\mathbb{R}_+$, $t\in[0,T]$, $x\in\mathbb{R}$, $K\in\mathbb{R}_+$, where $\mathscr{F}^{-1}$ denotes the improper inverse Fourier transform in the sense of Section B.1. From (3.1) it follows that $\Psi_t(T,u)=\Psi_T(T,u)$ for $T<t$. By (3.5, 3.6) this part $\Psi_t(T,u), T<t$ of the codebook does not affect option prices and is hence irrelevant.

Remark 3.3. The existence of these processes is implied by the assumptions above. Indeed, by Fubini's theorem for ordinary and stochastic integrals [32, Theorem IV.65], we have

$$\int_0^T |\Psi_t(r,u)|dr < \infty.$$

Fix $\omega\in\Omega$. Since $u\mapsto\int_t^T \Psi_t(r,u)(\omega)dr\in\Pi$, there is a random variable Y on some probability space $(\widetilde{\Omega},\widetilde{\mathscr{F}},\widetilde{P})$ which has infinitely divisible distribution and characteristic function

$$\widetilde{E}(e^{iuY}) = \exp\left(\int_t^T \Psi_t(r,u)(\omega)dr\right), \quad u\in\mathbb{R}.$$

Since this function is in Π, we have $\widetilde{E}(e^Y)=1$. Thus Proposition B.4 yields the existence of the inverse Fourier transform in Equation (3.5). Moreover, it implies $C_t(T,K)(\omega) = \widetilde{E}((S_t(\omega)e^Y-K)^+)$ and thus we have $0\le C_t(T,K)(\omega)\le S_t(\omega)$ and $0\le P_t(T,K)(\omega)\le K$ for any $K\in\mathbb{R}_+$, $T\in\mathbb{R}_+$, $t\in[0,T]$, where $P_t(T,K):=C_t(T,K)+K-S_t$ for any $K\in\mathbb{R}_+$, $T\in\mathbb{R}_+$, $t\in[0,T]$.

As noted above, we model asset prices under a risk-neutral measure for the whole market. Put differently, we are interested in risk-neutral option surface models in the following sense.

Definition 3.4. An option surface model $(X,\Psi_0,\alpha,\beta,M)$ is called *risk neutral* if the corresponding stock S and all European call options $C(T,K)$, $T\in\mathbb{R}_+$, $K>0$ are σ-martingales or, equivalently, local martingales (cf. [27, Proposition 3.1 and Corollary 3.1]). It is called *strongly risk neutral* if S and all $C(T,K)$ are martingales.

Below, risk-neutral option surface models are characterized in terms of the following properties.

Definition 3.5. An option surface model $(X,\Psi_0,\alpha,\beta,M)$ satisfies the *consistency condition* if

$$\psi_t^X(u) = \Psi_{t-}(t,u), \quad u\in\mathbb{R}$$

outside some $dP\otimes dt$-null set. Moreover, it satisfies the *drift condition* if

$$\left(u,-i\int_t^T \beta_t(r,u)dr\right)_{t\in\mathbb{R}_+} \in \mathscr{U}^{(X,M)}$$

and

$$\int_t^T \alpha_t(r,u)dr = \psi_t^X(u) - \psi_t^{(X,M)}\left(u,-i\int_t^T \beta_t(r,u)dr\right) \tag{3.7}$$

outside some $dP\otimes dt$-null set for any $T\in\mathbb{R}_+$, $u\in\mathbb{R}$. Finally, the option surface model satisfies the *conditional expectation condition* if

$$\exp\left(\int_t^T \Psi_t(r,u)dr\right)=E(e^{iu(X_T-X_t)}|\mathscr{F}_t)$$

for any $T\in\mathbb{R}_+$, $t\in[0,T]$, $u\in\mathbb{R}$.

Remark 3.6. The drift condition can be rewritten as

$$\alpha_t(T,u)=-\partial_T\left(\psi_t^{(X,M)}\left(u,-i\int_t^T\beta_t(r,u)dr\right)\right) \tag{3.8}$$

for almost all $T\in\mathbb{R}_+$. It gets even simpler if X and M are assumed to be locally independent in the sense of Definition A.10:

$$\alpha_t(T,u)=-\partial_T\left(\psi_t^{M}\left(-i\int_t^T\beta_t(r,u)dr\right)\right). \tag{3.9}$$

If the derivative $\psi_t'(u):=\partial_u\psi_t^M(u)$ exists as well, the drift condition simplifies once more and turns into

$$\alpha_t(T,u)=i\psi_t'\left(-i\int_t^T\beta_t(r,u)dr\right)\beta_t(T,u).$$

Now consider the situation that M is a one-dimensional Brownian motion which is locally independent of the return process X. Then $\psi^M(u)=-u^2/2$ and the drift condition reads as

$$\alpha_t(T,u)=-\beta_t(T,u)\int_t^T\beta_t(r,u)dr.$$

Thus the drift condition for option surface models is similar to the HJM drift condition (cf. [22]).

Drift condition (3.7) seems to rely on the joint exponent of X and M. However, (3.9) suggests that only partial knowledge about X is needed. It is in fact sufficient to specify the joint exponent $\psi^{(X^{\parallel},M)}$ of M and the dependent part $X^{\parallel}$ of X relative to M, which is defined in Section A.3. Using this notion, Equation (3.8) can be rewritten as

$$\alpha_t(T,u)=-\partial_T\left(\psi_t^{(X^{\parallel},M)}\left(u,-i\int_t^T\beta_t(r,u)dr\right)\right) \tag{3.10}$$

because $\psi^{(X,M)}=\psi^{(X^{\perp},0)}+\psi^{(X^{\parallel},M)}$ and the first summand on the right-hand side does not depend on its second argument.

3.2. **Necessary and sufficient conditions.** The goal of this section is to prove the following characterisation of risk-neutral option surface models.

Theorem 3.7. *For any option surface model $(X,\Psi_0,\alpha,\beta,M)$ the following statements are equivalent.*

(1) *It is strongly risk neutral.*
(2) *It is risk neutral.*
(3) *It satisfies the conditional expectation condition.*
(4) *It satisfies the consistency and drift conditions.*

The remainder of this section is devoted to the proof of Theorem 3.7. We proceed according to the following scheme

$$(1)\Rightarrow(2)\Rightarrow(3)\Rightarrow(4)\Rightarrow(3)\Rightarrow(1).$$

We use the notation

$$\begin{aligned}
\delta_t(T,u) &:= \int_t^T \alpha_t(r,u)dr - \psi_t^X(u), \\
\sigma_t(T,u) &:= \int_t^T \beta_t(r,u)dr, \\
\Gamma_t(T,u) &:= \int_0^T \Psi_0(r,u)dr + \int_0^t \delta_s(T,u)ds + \int_0^t \sigma_s(T,u)dM_s.
\end{aligned}$$

The existence of the integrals above is implied by the condition for option surface models. Observe that $\Gamma(T,u)$ is a semimartingale.

Lemma 3.8. *For any $u \in \mathbb{R}, T \in \mathbb{R}_+, t \in [0,T]$ we have*

$$\Gamma_t(T,u) - \Gamma_t(t,u) = \int_t^T \Psi_t(r,u)dr.$$

Proof. Using the definition of Γ, δ, σ and applying Fubini's theorem as in [32, Theorem IV.65] yields

$$\begin{aligned}
\Gamma_t(T,u) - \Gamma_t(t,u) &= \int_t^T \Psi_0(r,u)dr + \int_t^T \int_0^t \alpha_s(r,u)dsdr \\
&\quad + \int_t^T \int_0^t \beta_s(r,u)dM_sdr \\
&= \int_t^T \Psi_t(r,u)dr.
\end{aligned}$$

□

Lemma 3.9. *For any $u \in \mathbb{R}$, $t \in \mathbb{R}_+$ we have*

$$\Gamma_t(t,u) = \int_0^t \left(\Psi_{s-}(s,u) - \psi_s^X(u)\right) ds.$$

Proof. By Fubini's theorem for ordinary and stochastic integrals we have

$$\begin{aligned}
\Gamma_t(t,u) &= \int_0^t \Psi_0(r,u)dr + \int_0^t \delta_s(t,u)ds + \int_0^t \sigma_s(t,u)dM_s \\
&= \int_0^t \left(\Psi_0(r,u) + \int_0^r \alpha_s(r,u)ds - \psi_r^X(u)\right)dr + \int_0^t \int_0^r \beta_s(r,u)dM_sdr.
\end{aligned}$$

This yields the claim. □

Lemma 3.10. *If $(X, \Psi_0, \alpha, \beta, M)$ is risk neutral, then it satisfies the conditional expectation condition.*

Proof. Let $T \in \mathbb{R}_+$. We define

$$O_t(T,x) := \begin{cases} e^{-x}C_t(T,e^x) & \text{if } x \geq 0, \\ e^{-x}P_t(T,e^x) & \text{if } x < 0, \end{cases}$$

where $P_t(T,K) := C_t(T,K) + K - S_t$ for any $K \in \mathbb{R}_+$, $t \in [0,T], x \in \mathbb{R}$. Then we have

$$O_t(T,x) = \begin{cases} (e^{X_t - x} - 1)^+ + \mathscr{O}_t(T, x - X_t) & \text{if } x \geq 0, \\ (1 - e^{X_t - x})^+ + \mathscr{O}_t(T, x - X_t) & \text{if } x < 0. \end{cases}$$

We calculate the Fourier transform of $O_t(T,x)$ in two steps by considering the summands separately. The improper Fourier transform of the second summand $\mathscr{O}_t(T,x-X_t)$ exists and satisfies

$$\begin{aligned}\mathscr{F}\{x\mapsto \mathscr{O}_t(T,x-X_t)\}(u) &= \mathscr{F}\{x\mapsto \mathscr{O}_t(T,x)\}(u)e^{iuX_t}\\ &= \frac{1-\exp\left(\int_t^T \Psi_t(r,u)dr\right)}{u^2+iu}e^{iuX_t}\end{aligned}$$

for any $u\in\mathbb{R}\setminus\{0\}$ by Remark 3.3, Proposition B.4 and the translation property for the Fourier transform, which holds for the improper Fourier transform as well. The Fourier transform of the first summand $A_t(T,x) := O_t(T,x) - \mathscr{O}_t(T,x-X_t)$ exists and equals

$$\mathscr{F}\{x\mapsto A_t(T,x)\}(u) = \frac{1}{iu} - \frac{e^{X_t}}{iu-1} - \frac{e^{iuX_t}}{u^2+iu}$$

for any $u\in\mathbb{R}\setminus\{0\}$. Therefore the improper Fourier transform of $x\mapsto O_t(T,x)$ exists and is given by

$$\mathscr{F}\{x\mapsto O_t(T,x)\}(u) = \frac{1}{iu} - \frac{e^{X_t}}{iu-1} - \frac{\exp(iuX_t+\int_t^T \Psi_t(r,u)dr)}{u^2+iu}, \tag{3.11}$$

for any $u\in\mathbb{R}\setminus\{0\}$. By Lemmas 3.8 and 3.9 we have that the right-hand side of (3.11) is a semimartingale, in particular it has càdlàg paths. Remark 3.3 yields that $0\le P_t(T,K)\le K$. Hence $(P_t(T,K))_{t\in[0,T]}$ is a martingale because it is a bounded local martingale. Let $(\tau_n)_{n\in\mathbb{N}}$ denote a common localising sequence for $(C_t(T,1))_{t\in[0,T]}$ and S, i.e. S^{τ_n}, $C^{\tau_n}(T,1)$ are uniformly integrable martingales for any $n\in\mathbb{N}$. Since $C_t^{\tau_n}(T,K)\le C_t^{\tau_n}(T,1)$ for $K\in[1,\infty)$, we have that $(\tau_n)_{n\in\mathbb{N}}$ is a common localising sequence for all European calls with maturity T and strike $K\ge 1$. The definition of $O_t(T,x)$ yields that it is a local martingale for any $x\in\mathbb{R}$ and $(\tau_n)_{n\in\mathbb{N}}$ is a common localising sequence for $(O_t(T,x))_{t\in[0,T]}, x\in\mathbb{R}$.

Fix $\omega\in\Omega$. Since $u\mapsto \int_t^T \Psi_t(r,u)(\omega)dr$ is in Π for any $t\in[0,T]$, there is a random variable Y on some space $(\widetilde{\Omega},\widetilde{\mathscr{F}},\widetilde{P})$ with characteristic function

$$\widetilde{E}(e^{iuY}) = \exp\left(\int_t^T \Psi_t(r,u)(\omega)dr\right).$$

Then $\widetilde{E}(e^Y)=1$ and

$$O_t(T,x)(\omega) = \begin{cases} \widetilde{E}((S_t(\omega)e^{Y-x}-1)^+) & \text{if } x\ge 0,\\ \widetilde{E}(1-S_t(\omega)e^{Y-x})^+) & \text{if } x<0,\end{cases}$$

cf. Remark 3.3. By Corollary B.3 we have $|\int_{-C}^{\infty} e^{iux}O_t(T,x)dx| \le S_t(\omega)+\frac{1+2|u|}{u^2}$. Proposition B.5 yields that

$$(\mathscr{F}\{x\mapsto O_t(T,x)\}(u))_{t\in[0,T]}$$

and hence $(\Phi_t(u))_{t\in[0,T]}$ given by

$$\Phi:\Omega\times[0,T]\times\mathbb{R}\to\mathbb{C},\quad (\omega,t,u)\mapsto \exp\left(iuX_t(\omega)+\int_t^T \Psi_t(r,u)(\omega)dr\right)$$

are local martingales for any $u\in\mathbb{R}\setminus\{0\}$. Since $u\mapsto \int_t^T \Psi_t(r,u)(\omega)dr$ is in Π for any $t\in[0,T]$, $\omega\in\Omega$, its real part is bounded by 0 from above, cf. Lemma A.14. Hence $|\Phi_t(u)|\le 1$ and thus $(\omega,t)\mapsto \Phi_t(u)(\omega)$ is a true martingale for any $u\in\mathbb{R}\setminus\{0\}$. By $\Phi_t(0)=1$ it is a martingale for $u=0$ as well. Since $\Phi_T(u)=\exp(iuX_T)$, the two martingales $(\Phi_t(u))_{t\in[0,T]}$ and $(E(\exp(iuX_T)|\mathscr{F}_t))_{t\in[0,T]}$ coincide for any $u\in\mathbb{R}$. Thus we have

$$\exp\left(\int_t^T \Psi_t(r,u)dr\right)=\exp(-iuX_t)\Phi_t(u)=E(e^{iu(X_T-X_t)}|\mathscr{F}_t)$$

for any $u\in\mathbb{R}$, $t\in[0,T]$. □

Lemma 3.11 (Drift condition in terms of δ and σ)**.** *If $(X,\Psi_0,\alpha,\beta,M)$ satisfies the conditional expectation condition, we have the drift condition*

$$\delta_t(T,u)=\Psi_{t-}(t,u)-\psi_t^X(u)-\psi_t^{(X,M)}(u,-i\sigma_t(T,u))$$

outside some $dP\otimes dt$-null set for $T\in\mathbb{R}_+$, $u\in\mathbb{R}$. In particular, $(u,-i\sigma(T,u))\in\mathscr{U}^{(X,M)}$.

Proof. For $u\in\mathbb{R}$ and $T\in\mathbb{R}_+$ define the process $Z_t:=iuX_t+\int_t^T\Psi_t(r,u)dr$. The conditional expectation condition yields that $\exp(Z_t)=E(e^{iuX_T}|\mathscr{F}_t)$ is a martingale. Hence $-i\in\mathscr{U}^Z$ and $\psi_t^Z(-i)=0$ by Lemma A.16. With $Y_t:=\Gamma_t(t,u)$ we obtain

$$\begin{aligned}
0 &= \psi_t^Z(-i)\\
&= \psi_t^{iuX+\Gamma(T,u)-Y}(-i)\\
&= \psi_t^{iuX+\Gamma(T,u)}(-i)-\left(\Psi_{t-}(t,u)-\psi_t^X(u)\right)\\
&= \psi_t^{(iuX,\Gamma(T,u))}(-i,-i)-\Psi_{t-}(t,u)+\psi_t^X(u)\\
&= \psi_t^{(X,M)}(u,-i\sigma_t(T,u))+\delta_t(T,u)-\Psi_{t-}(t,u)+\psi_t^X(u),
\end{aligned}$$

where the second equation follows from Lemma 3.8, the third from Lemmas 3.9 and A.20, the fourth from Lemma A.19 and the last from Lemmas A.18 and A.20. □

Corollary 3.12 (Consistency condition)**.** *If $(X,\Psi_0,\alpha,\beta,M)$ satisfies the conditional expectation condition, then it satisfies the consistency condition.*

Proof. Lemma 3.11 and the definition of δ yield

$$\Psi_{t-}(t,u)=\delta_t(t,u)+\psi_t^X(u)+\psi_t^{(X,M)}(u,0)=\psi_t^X(u).$$

□

Corollary 3.13 (Drift condition)**.** *If $(X,\Psi_0,\alpha,\beta,M)$ satisfies the conditional expectation condition, then it satisfies the drift condition.*

Proof. This follows from Lemma 3.11 and Corollary 3.12. □

Lemma 3.14. *If the option surface model satisfies the consistency condition, then*

$$\Gamma_t(T,u)=\int_t^T\Psi_t(r,u)dr$$

for any $T\in\mathbb{R}_+$, $t\in[0,T]$, $u\in\mathbb{R}$.

Proof. This is a direct consequence of Lemmas 3.8 and 3.9. □

Lemma 3.15. *If the option surface model satisfies the drift condition, then*

$$\Phi_t(T,u):=\exp(iuX_t+\Gamma_t(T,u))$$

defines a local martingale $(\Phi_t(T,u))_{t\in[0,T]}$ for any $u\in\mathbb{R}$, $T\in\mathbb{R}_+$.

Proof. Fix T,u and define $Z_t := iuX_t + \Gamma_t(T,u)$. By the drift condition and Lemmas A.18 – A.20 we have

$$\begin{aligned} 0 &= \psi_t^{(X,M)}(u,-i\sigma_t(T,u)) + \delta_t(T,u) \\ &= \psi_t^{(X,\sigma(T,u)\bullet M)}(u,-i) + \delta_t(T,u) \\ &= \psi_t^{(iuX,\Gamma(T,u))}(-i,-i) \\ &= \psi_t^{iuX+\Gamma(T,u)}(-i) \\ &= \psi_t^{Z}(-i). \end{aligned}$$

Hence $\exp(Z)$ is a local martingale by Lemma A.16. □

Lemma 3.16. $(X,\Psi_0,\alpha,\beta,M)$ *satisfies the drift and consistency conditions if and only if it satisfies the conditional expectation condition.*

Proof. $\Leftarrow$: This is a restatement of Corollaries 3.13 and 3.12.
$\Rightarrow$: Fix $u \in \mathbb{R}$, $T \in \mathbb{R}_+$. Lemma A.14 implies that the absolute value of

$$\Phi_t(T,u) := \exp\left(iuX_t + \int_t^T \Psi_t(r,u)dr\right)$$

is bounded by 1. By Lemmas 3.14 and 3.15, $\Phi(T,u)$ is a local martingale and hence a martingale. This yields

$$\Phi_t(T,u) = E(\Phi_T(T,u)|\mathscr{F}_t) = E(e^{iuX_T}|\mathscr{F}_t).$$

□

Lemma 3.17. *If the option surface model* $(X,\Psi_0,\alpha,\beta,M)$ *satisfies the conditional expectation condition, then* $S = e^X$ *is a martingale.*

Proof. For $T \in \mathbb{R}_+$, $t \in [0,T]$ we have

$$E(e^{iu(X_T-X_t)}|\mathscr{F}_t) = \exp\left(\int_t^T \Psi_t(r,u)dr\right).$$

Since $u \mapsto \int_t^T \Psi_t(r,u)(\omega)dr$ is in Π, we have $E(e^{X_T-X_t}|\mathscr{F}_t) = 1$, cf. Remark 3.3. □

Lemma 3.18. *If the option surface model* $(X,\Psi_0,\alpha,\beta,M)$ *satisfies the conditional expectation condition, it is strongly risk neutral.*

Proof. Lemma 3.17 implies that e^X is a martingale and in particular that e^{X_t} is integrable for any $t \in \mathbb{R}_+$. Let $T \in \mathbb{R}_+$, $t \in [0,T]$. We define

$$\begin{aligned} \widetilde{C}(K) &:= E((e^{X_T}-K)^+|\mathscr{F}_t), \\ \widetilde{\mathscr{O}}(x) &:= e^{-(x+X_t)}\widetilde{C}(e^{x+X_t}) - (e^{-x}-1)^+, \\ Y &:= X_T - X_t \end{aligned}$$

for any $K \in \mathbb{R}_+$. Obviously we have

$$\widetilde{\mathscr{O}}(x) = \begin{cases} E((e^{Y-x}-1)^+|\mathscr{F}_t) & \text{if } x \geq 0, \\ E((1-e^{Y-x})^+|\mathscr{F}_t) & \text{if } x < 0 \end{cases}$$

and $E(e^Y|\mathscr{F}_t)=1$. Hence Corollary B.4, the conditional expectation condition, and the definition of $\mathscr{O}$ yield

$$\begin{aligned}\mathscr{F}\{x\mapsto\widetilde{\mathscr{O}}(x)\}(u) &= \frac{1-E(e^{iuY}|\mathscr{F}_t)}{u^2+iu}\\ &= \frac{1-\exp\left(\int_t^T\Psi_t(r,u)dr\right)}{u^2+iu}\end{aligned}$$

as well as

$$\begin{aligned}\widetilde{\mathscr{O}}(x) &= \mathscr{F}^{-1}\left\{u\mapsto\frac{1-\exp\left(\int_t^T\Psi_t(r,u)dr\right)}{u^2+iu}\right\}(x)\\ &= \mathscr{O}_t(T,x).\end{aligned}$$

Thus we have

$$\widetilde{C}(K)=C_t(T,K)$$

for any $K\in\mathbb{R}_+$. Consequently, the option surface model $(X,\Psi_0,\alpha,\beta,M)$ is strongly risk neutral. □

Proof of Theorem 3.7. $(1)\Rightarrow(2)$ is obvious.
$(2)\Rightarrow(3)$ has been shown in Lemma 3.10.
$(3)\Leftrightarrow(4)$ is the conclusion of Lemma 3.16.
$(3)\Rightarrow(1)$ has been shown in Lemma 3.18.

3.3. **Musiela parametrisation.** In practice one may prefer to parametrise the codebook in terms of time-to-maturity $x:=T-t$ instead of maturity T, which is referred to as *Musiela parametrisation* in interest rate theory. However, in order to express the corresponding codebook dynamics, we need some additional regularity.

Proposition 3.19. *Let $(X,\Psi_0,\alpha,\beta,M)$ be an option surface model such that*

(1) *M is of the form $M_t=N_t+\int_0^t v_s ds$ with some locally square-integrable martingale N and an integrable predictable process v,*
(2) *$T\mapsto\alpha_t(T,u),\beta_t(T,u),\Psi_0(T,u)$ are continuously differentiable for any $t\in\mathbb{R}_+$, $u\in\mathbb{R}$,*
(3) *$\int_0^T|\partial_r\beta_t(r,u)|^2dr<\infty$ for any $t,T\in\mathbb{R}_+$, $u\in\mathbb{R}$,*
(4) *$\int_0^t\sup_{r\in[0,T]}|\partial_r\alpha_s(r,u)+\partial_r\beta_s(r,u)v_s|ds<\infty$ for any $t,T\in\mathbb{R}_+$, $u\in\mathbb{R}$,*
(5) *$((\int_0^T(|\beta_t^j(r,u)|^2+|\partial_r\beta_t^j(r,u)|^2)dr)^{1/2})_{t\in\mathbb{R}_+}\in L^2_{\mathrm{loc}}(N^j)$ for any $T\in\mathbb{R}_+$, $u\in\mathbb{R}$, $j\in\{1,\dots,d\}$, where $L^2_{\mathrm{loc}}(N^j)$ is defined as in* [25, I.4.39].

Define $\check{\alpha}_t(x,u):=\alpha_t(t+x,u)$, $\check{\beta}_t(x,u):=\beta_t(t+x,u)$, $\check{\Psi}_t(x,u):=\Psi_t(t+x,u)$ for any $t\in\mathbb{R}_+$, $x\in\mathbb{R}_+$, $u\in\mathbb{R}$. For any fixed $u\in\mathbb{R}$, the mapping $x\mapsto\check{\Psi}_t(x,u)$ is differentiable for dt-almost all $t\in\mathbb{R}_+$ and we have

$$\check{\Psi}_t(x,u)=\check{\Psi}_0(x,u)+\int_0^t\left(\check{\alpha}_s(x,u)+\partial_x\check{\Psi}_s(x,u)\right)ds+\int_0^t\check{\beta}_s(x,u)M_s$$

for any $t\in\mathbb{R}_+$, $x\in\mathbb{R}$, $u\in\mathbb{R}$.

Proof. Since

$$\begin{aligned}\Psi_t(T,u) &= \Psi_0(T,u)+\int_0^{t\wedge T}\alpha_s(T,u)ds+\int_0^{t\wedge T}\beta_s(T,u)dM_s\\ &= \Psi_0(T,u)+\int_0^{t\wedge T}(\alpha_s(T,u)+\beta_s(T,u)v_s)ds+\int_0^{t\wedge T}\beta_s(T,u)dN_s,\end{aligned}$$

we can assume w.l.o.g. that M is a locally square-integrable martingale. By localisation, it even suffices to consider square-integrable martingales.

By Fubini's theorem for the stochastic integral (cf. [32, Theorem IV.65]), we have

$$\begin{aligned}
\check{\Psi}_t(x,u) &= \check{\Psi}_0(t+x,u)+\int_0^t \check{\alpha}_s(t-s+x,u)ds+\int_0^t \check{\beta}_s(t-s+x,u)dM_s \\
&= \check{\Psi}_0(x,u)+\int_0^t \check{\alpha}_s(x,u)ds+\int_0^t \check{\beta}_s(x,u)dM_s \\
&\quad +\int_0^t \partial_x\check{\Psi}_0(r+x,u)dr+\int_0^t\int_s^t \partial_x\check{\alpha}_s(r-s+x,u)drds \\
&\quad +\int_0^t\int_s^t \partial_x\check{\beta}_s(r-s+x,u)drdM_s \\
&= \check{\Psi}_0(x,u)+\int_0^t \check{\alpha}_s(x,u)ds+\int_0^t \check{\beta}_s(x,u)dM_s \\
&\quad +\int_0^t \partial_x\check{\Psi}_0(x+r,u)dr+\int_0^t\int_0^r \partial_x\alpha_s(r+x,u)dsdr \\
&\quad +\int_0^t\int_0^r \partial_x\beta_s(r+x,u)dM_sdr \\
&= \check{\Psi}_0(x,u)+\int_0^t \check{\alpha}_s(x,u)ds+\int_0^t \check{\beta}_s(x,u)dM_s \\
&\quad +\int_0^t \Big(\partial_x\check{\Psi}_0(x+r,u)+\partial_x\int_0^r \alpha_s(r+x,u)ds \\
&\quad +\partial_x\int_0^r \beta_s(r+x,u)dM_s\Big)dr \\
&= \check{\Psi}_0(x,u)+\int_0^t \big(\check{\alpha}_s(x,u)+\partial_x\check{\Psi}_s(x,u)\big)\,ds+\int_0^t \check{\beta}_s(x,u)M_s
\end{aligned}$$

for any $t,x\in\mathbb{R}_+$, $u\in\mathbb{R}$ where the fourth equality is explained below. For fixed $t,x\in\mathbb{R}_+$, $u\in\mathbb{R}$ define Hilbert spaces $H:=L^2([0,t+x],\mathbb{R})$ and

$$H_1 := \{f\in H : f \text{ is differentiable and } f'\in H\}$$

with norm

$$\|f\|_{H_1} := \sqrt{\|f\|_H^2+\|f'\|_H^2}.$$

The mapping $r\mapsto\beta_t(r+x,u)$ is in H_1 by assumption. Since $\partial_r : H_1\to H$, $f\mapsto f'$ is linear and continuous, [31, Theorem 8.7(v)] yields

$$\int_0^t \partial_r\beta_s(\cdot,u)dM_s = \partial_r\int_0^t \beta_s(\cdot,u)dM_s$$

and hence

$$\int_0^t\int_0^r \partial_x\beta_s(r+x,u)dM_sdr = \int_0^t \partial_x\int_0^r \beta_s(r+x,u)dM_sdr.$$

□

4. Constructing models from building blocks

In this section we turn to existence and uniqueness results for option surface models which are driven by a subordinator M.

4.1. **Building blocks.** Theorem 3.7 indicates thet neither the drift part α of the codebook nor the dynamics of the return process X can be chosen arbitrarily if one wants to end up with a risk-neutral option surface model. What ingredients do we need in order to construct such a model? It seems natural to consider volatiliy processes β that are functions of the present state of the codebook, i.e.

$$\beta_t(T,u)(\omega) = b(t,\Psi_{t-}(\cdot,\cdot)(\omega))(T,u)$$

for some deterministic function $b : \mathbb{R}_+ \times L^1(\mathbb{R}_+,\Pi) \to L^1(\mathbb{R}_+,\Pi)$, where $L^1(\mathbb{R}_+,\Pi)$ denotes some suitable space of conceivable codebook states, i.e. essentially of functions $\mathbb{R}_+ \to \Pi$. It is specified below. In order to hope for uniqueness, we need to fix the initial values X_0 and $\Psi_0(\cdot,\cdot)$, function b and the law of the driving process M. The drift part in (3.1) need not be specified as it is implied by the drift condition. But we need some information on X. Although its dynamics seem to be determined by the consistency condition, the joint behaviour of X and M is not. The latter, however, is needed for the drift condition (3.7) resp. (3.8). In view of (3.10), we assume that the joint law of M and the dependent part $X^\parallel$ of X relative to M in the sense of Section A.3 are given. More specifically, we suppose that $(X^\parallel, M)$ is a Lévy process with given Lévy exponent $\psi^{(X^\parallel,M)} = \gamma$. The components of M are supposed to be subordinators. Altogether, we suggest to construct models based on a quadrupel (x_0,ψ_0,b,γ), where $x_0 \in \mathbb{R}$ and $\psi_0 \in L^1(\mathbb{R}_+,\Pi)$ stand for the initial states of the return process and the codebook, respectively.

In order to derive existence and uniqueness results, we still need to specify the domain and codomain of b. For ease of notation, we focus on one-dimensional driving processes M. The vector-valued case can be treated along the same lines.

Let E denote the set of continuous functions $\psi : \mathbb{R} \to \mathbb{C}$ and

$$\|\psi\|_m := \sup\{|\psi(u)| : |u| \leq m\}$$

for $m \in \mathbb{R}_+$. By $\mathscr{L}^1(\mathbb{R}_+,E)$ we denote the set of measurable functions $\psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ such that $\psi(T,\cdot) \in E$ and

$$\|\psi\|_{T,m} := \int_0^T \|\psi(r,\cdot)\|_m dr < \infty$$

for any $T,m \in \mathbb{R}_+$. For $\psi \in \mathscr{L}^1(\mathbb{R}_+,E)$ we set

$$[\psi] := \{\varphi \in \mathscr{L}^1(\mathbb{R}_+,E) : \psi(T,\cdot) = \varphi(T,\cdot) \text{ for almost any } T \in \mathbb{R}_+\}.$$

Moreover, we define the space

$$L^1(\mathbb{R}_+,E) := \{[\psi] : \psi \in \mathscr{L}^1(\mathbb{R}_+,E)\}.$$

as usual. Finally, we set

$$L^1(\mathbb{R}_+,\Pi) \quad := \quad \left\{ \psi \in L^1(\mathbb{R}_+,E) : \int_t^T \psi(r,\cdot)dr \in \Pi \text{ for any } 0 \leq t \leq T < \infty \right\},$$

where we refer to the Bochner integral in the sense of Definition C.6 and Example C.2.

Lemma 4.1. *The following statements hold:*

(1) $(E, \|\cdot\|_m)$ *is a complete and separable semi-normed space for any* $m \in \mathbb{R}_+$.
(2) $(L^1(\mathbb{R}_+,E), \|\cdot\|_{T,m})$ *is a complete and separable semi-normed space for any* $T,m \in \mathbb{R}_+$. *If* $x \in L^1(\mathbb{R}_+,E)$ *with* $\|x\|_{n,n} = 0$ *for any* $n \in \mathbb{N}$, *we have* $x = 0$. *Moreover, if* $(x_k)_{k\in\mathbb{N}}$ *is a Cauchy sequence in* $L^1(\mathbb{R}_+,E)$ *relative to any* $\|\cdot\|_{n,n}$, $n \in \mathbb{N}$, *there exists*

$x \in L^1(\mathbb{R}_+, E)$ such that $\lim_{k\to\infty} \|x_k - x\|_{n,n} = 0$, $n \in \mathbb{N}$. Consequently, $(L^1(\mathbb{R}_+, E), d)$ is a separable Fréchet space for the metric

$$d(\psi, \varphi) := \sum_{n\in\mathbb{N}} 2^{-n}(1 \wedge \|\psi - \varphi\|_{n,n}), \quad \psi, \varphi \in L^1(\mathbb{R}_+, E).$$

(3) *$\Pi \subset E$ is a convex cone. If A is a Borel subset of $\mathbb{R}$, μ a finite measure on A, and $\psi : A \times \mathbb{R} \to \mathbb{C}$ is measurable with $\psi(r, \cdot) \in \Pi$ and $\int_A \|\psi(r, \cdot)\|_m \mu(dr) < \infty$ for all $m \in \mathbb{N}$, then the mapping $u \mapsto \int_A \psi(r, u)\mu(dr)$ is in Π.*
(4) *If $\psi \in L^1(\mathbb{R}_+, E)$ and $\psi(T, \cdot) \in \Pi$ for almost all $T \in \mathbb{R}_+$, then $\psi \in L^1(\mathbb{R}_+, \Pi)$.*
(5) *For any increasing function $X : \mathbb{R}_+ \to \mathbb{R}_+$ and any locally X-integrable function $\eta : \mathbb{R}_+ \to L^1(\mathbb{R}_+, \Pi)$, we have $\int_0^t \eta_s dX_s \in L^1(\mathbb{R}_+, \Pi)$ for any $t \in \mathbb{R}_+$. Here, we refer to Bochner integration on $L^1(\mathbb{R}_+, E)$, cf. Example C.2 and Definition C.6.*
(6) *Π is a Borel subset of E and consequently $L^1(\mathbb{R}_+, \Pi)$ is a Borel subset of $L^1(\mathbb{R}_+, E)$ (relative to the Borel-σ-field generated by the metric d).*

Proof. (1) This follows from the fact that the continuous functions on $[-m, m]$ are a separable Banach space relative to the uniform norm.

(2) $(L^1(\mathbb{R}_+, E), \|\cdot\|_{T,m})$ is a complete and separable semi-normed space because the corresponding Lebesgue-Bochner space of integrable functions on $[0, T]$ with values in the Banach space of continuous functions $[-m, m] \to \mathbb{C}$ is a Banach space.

Let Q be a countable dense set in E. Define

$$S := \left\{ \sum_{j=1}^{n} q_j 1_{(a_j, b_j]} : n \in \mathbb{N}, q \in Q^n, a, b \in \mathbb{Q}^n \right\}.$$

S is dense in $L^1(\mathbb{R}_+, E)$ because S is obviously dense in

$$T := \left\{ \sum_{j=1}^{n} q_j 1_{A_j} : n \in \mathbb{N}, q \in Q^n, A_1, \dots, A_n \in \mathscr{B}(\mathbb{R}_+) \right\}$$

and T is dense in $L^1(\mathbb{R}_+, E)$, cf. Section C.1 of the appendix. This shows separability of $(L^1(\mathbb{R}_+, E), \|\cdot\|_{T,m})$. The remaining statements are straightforward.

(3) Π is obviously a convex cone. Define

$$\Psi : \mathbb{R} \to \mathbb{C}, \quad u \mapsto \int_A \psi(r, u)\mu(dr).$$

Then Ψ is continuous and it is the pointwise limit of Lévy exponents. Hence Lévy's continuity theorem (see [30, Theorem 3.6.1]) together with [30, Theorem 5.3.3] yield that Ψ is the characteristic exponent of an infinitely divisible random variable X.

Fix the truncation function $h : \mathbb{R} \to \mathbb{R}, x \mapsto x1_{\{|x|\le 1\}}$. For all $r \in A$ let (b_r, c_r, F_r) be the Lévy-Khintchine triplet corresponding to $\psi(r, \cdot)$. A detailed analysis of the proof of [25, Lemma II.2.44] yields integrability of b and c and that F is a transition kernel satisfying $\int_A \int (|x|^2 \wedge 1)F_r(dx)\mu(dr) < \infty$.

In order to prove $\Psi \in \Pi$ we have to show that $Ee^X = 1$. Let (B, C, ν) be the triplet corresponding to Ψ and h. Then [25, Theorem II.4.16] yields

$$\exp(\Psi(u)) = \exp\left(iuB - \frac{u^2}{2}C + \int_{\mathbb{R}} (e^{iux} - 1 - iuh(x))\nu(dx) \right)$$

as well as

$$\begin{aligned}
&\exp(\Psi(u)) \\
= &\exp\left(\int_A \psi(r,u)\mu(dr)\right) \\
= &\exp\left(iu\int_A b_r dr - \frac{u^2}{2}\int_A c_r dr + \int_A\int (e^{iux}-1-iuh(x))F_r(dx)\mu(dr)\right)
\end{aligned}$$

for any $u \in \mathbb{R}$. From [25, Lemma II.2.44] we obtain $B = \int_A b_r\mu(dr)$, $C = \int_A c_r\mu(dr)$, and $\nu(G) = \int_A F_r(G)\mu(dr)$ for $G \in \mathscr{B}$. Consequently, we have

$$\begin{aligned}
E(e^X) &= \exp\left(B + \frac{1}{2}C + \int (e^x - 1 - h(x))\nu(dx)\right) \\
&= \exp\left(\int_A \left(b_r + \frac{1}{2}c_r + \int_{(-\infty,1]}(e^x-1-h(x))F_r(dx)\right)\mu(dr)\right. \\
&\qquad \left. + \int_{(1,\infty)}(e^x-1)\nu(dx)\right).
\end{aligned}$$

Tonelli's theorem yields $\int_{(1,\infty)}(e^x-1)\nu(dx) = \int_A\int_{(1,\infty)}(e^x-1)F_r(dx)\mu(dr)$. Hence

$$E(e^X) = \exp\left(\int_A \left(b_r + \frac{1}{2}c_r + \int (e^x-1-h(x))F_r(dx)\right)\mu(dr)\right) = 1.$$

(4) This is a consequence of Statement 3.
(5) This is a consequence of Statement 3 as well.
(6) E is a metric space relative to

$$\delta(\psi,\varphi) := \sum_{n\in\mathbb{N}} 2^{-n}(1 \wedge \|\psi-\varphi\|_n).$$

Let $C \subset E$ denote the set of all Lévy exponents. Lévy's continuity theorem (cf. [30, Theorem 3.6.1]) and [30, Theorem 5.3.3] imply that C is closed in E and in particular a Borel subset in E. The function

$$f : C \to \overline{\mathbb{R}}_+, \quad \varphi \mapsto \frac{1}{2\pi}\int_{-\infty}^{\infty} e^x \int_{-\infty}^{\infty} \exp\left(\varphi(u) - \frac{1}{2}(iu+u^2) - iux\right) du dx$$

is well defined and measurable, where $\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{\infty\}$. Indeed, observe that for $\varphi \in C$ the function

$$u \mapsto \varphi(u) - \frac{1}{2}(iu+u^2)$$

is a Lévy exponent and thus [30, Theorem 3.2.2] yields that

$$x \mapsto \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(\varphi(u) - \frac{1}{2}(iu+u^2) - iux\right) du$$

is a density function.

Let L be a Lévy process with Lévy exponent $\varphi \in C$ and W be an independent Brownian motion with diffusion coefficient 1 and drift rate $-1/2$. [30, Theorem 3.2.2] yields that

$$p : \mathbb{R} \to \mathbb{R}_+, \quad x \mapsto \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(\varphi(u) - \frac{1}{2}(iu+u^2) - iux\right) du$$

is the density function of L_1+W_1. Thus we have

$$E(e^{L_1}) \quad = \quad E(e^{L_1+W_1}) = \int_{-\infty}^{\infty} e^x p(x)dx = f(\varphi).$$

Hence $\Pi = f^{-1}(1)$, which implies that Π is measurable in E. It remains to be shown that $L^1(\mathbb{R}_+,\Pi)$ is a measurable set in $L^1(\mathbb{R}_+,E)$. For $a,b\in\mathbb{R}_+$ define the continuous and hence measurable map

$$I_{a,b}: L^1(\mathbb{R}_+,E)\to E, \quad \psi \mapsto \int_a^b \psi(r,\cdot)dr.$$

We obviously have

$$L^1(\mathbb{R}_+,\Pi)\subset M := \bigcap\{I^{-1}_{q_1,q_2}(\Pi) : q_1,q_2\in\mathbb{Q}_+, q_1\leq q_2\}.$$

We show that the two sets are in fact equal. Let $\psi\in M$ and $t,T\in\mathbb{R}_+$ with $t<T$. Then $\psi\in L^1(\mathbb{R}_+,E)$ and hence $\int_t^T\psi(r,\cdot)dr\in E$. Let $(q_n)_{n\in\mathbb{N}},(p_n)_{n\in\mathbb{N}}$ be sequences of rational numbers such that $q_n\downarrow t$, $p_n\uparrow T$, and $q_n\leq p_n$. Then

$$I_{t,T}\psi = I_{q_0,p_0}\psi + \sum_{n\in\mathbb{N}}(I_{q_{n+1},p_{n+1}} - I_{q_n,p_n})\psi.$$

Defining the finite measure μ on $\mathbb{N}$ by $\mu(\{n\}) := 1/n^2$ and the function

$$\gamma:\mathbb{N}\to\Pi, \quad n\mapsto n^2(I_{q_{n+1},p_{n+1}} - I_{q_n,p_n})\psi,$$

we obtain

$$I_{t,T}\psi = I_{q_0,p_0}\psi + \int_{\mathbb{N}}\gamma d\mu.$$

Hence Statement 3 yields $I_{t,T}\psi\in\Pi$, which in turn implies $\psi\in L^1(\mathbb{R}_+,\Pi)$.

□

We are now ready to formalise the notion of building blocks.

Definition 4.2. We call a quadruple (x_0,ψ_0,b,γ) *building blocks* of an option surface model if

(1) $x_0\in\mathbb{R}_+$,
(2) $\psi_0\in L^1(\mathbb{R}_+,\Pi)$,
(3) $b:\mathbb{R}_+\times L^1(\mathbb{R}_+,E)\to L^1(\mathbb{R}_+,E)$ is measurable (relative to the σ-fields $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{B}(L^1(\mathbb{R}_+,E))$ and $\mathscr{B}(L^1(\mathbb{R}_+,E))$, where $\mathscr{B}(L^1(\mathbb{R}_+,E))$ denotes the Borel-σ-field on the metric space $(L^1(\mathbb{R}_+,E),d)$ as introduced in Lemma 4.1),
(4) b maps $\mathbb{R}_+\times L^1(\mathbb{R}_+,\Pi)$ on a subset of $L^1(\mathbb{R}_+,\Pi)$,
(5) b is *locally Lipschitz* in the sense that for any $T\in\mathbb{R}_+$ there are $T_0,m_0\in\mathbb{R}_+$ such that for any $\widetilde{T}\geq T_0$, $m\geq m_0$ there exists $c\in\mathbb{R}_+$ such that
$$\|b(t,\psi_1)-b(t,\psi_2)\|_{\widetilde{T},m}\leq c\|\psi_1-\psi_2\|_{\widetilde{T},m}$$
holds for any $\psi_1,\psi_2\in L^1(\mathbb{R}_+,E)$ and any $t\in[0,T]$,
(6) $b(t,\psi)(T,u)=0$ for any $\psi\in L^1(\mathbb{R}_+,E)$, $u\in\mathbb{R}$ and any $t,T\in\mathbb{R}_+$ with $t>T$,
(7) $\sup_{t\in[0,T]}\|b(t,0)\|_{T,m}<\infty$ for any $T,m\in\mathbb{R}_+$,
(8) $\gamma:\mathbb{R}\times(\mathbb{R}+i\mathbb{R}_+)\to\mathbb{C}$ is the extended Lévy exponent on $\mathbb{R}\times(\mathbb{R}+i\mathbb{R}_+)$(cf. Section A.2) of an $\mathbb{R}^{1+1}$-dimensional Lévy process $(X^{\|},M)$ such that
   (a) M is a pure jump subordinator, i.e. $M_t=\sum_{s\leq t}\Delta M_s$, $t\in\mathbb{R}_+$,
   (b) $X^{\|}$ is the dependent part of $X^{\|}$ relative to M,
   (c) γ is differentiable, and
   (d) $\partial_2\gamma:\mathbb{R}\times(\mathbb{R}+i\mathbb{R}_+)\to\mathbb{C}$ is bounded and Lipschitz continuous.

Remark 4.3. By Remark B.11 the smoothness conditions (c,d) on γ are satisfied if both $X^{\parallel}$ and M have finite second moments.

Our goal is to find corresponding risk-neutral option surface models in the following sense.

Definition 4.4. An option surface model $(X,\Psi_0,\alpha,\beta,M)$ is said to be *compatible* with building blocks (x_0,ψ_0,b,γ) if

- $X_0 = x_0$,
- $\Psi_0 = \psi_0$,
- $\mathbb{R}_+ \to L^1(\mathbb{R}_+,E)$, $t \mapsto \Psi_t(\omega)$ is a well-defined, a.s. càdlàg mapping,
- $\beta_t(\omega) = b(t,\Psi_{t-}(\omega))$ for $dP\otimes dt$-almost any $(\omega,t)\in\Omega\times\mathbb{R}_+$,
- $\psi^{(X^{\parallel},M)} = \gamma$ on $\mathbb{R}\times(\mathbb{R}+i\mathbb{R}_+)$, where $X^{\parallel}$ denotes the dependent part of X relative to M.

Remark 4.5. In practice one may be interested in coefficients b of the form

$$b(t,\psi)(T,u) := \begin{cases} \check{b}(\psi^{(t)})(T-t,u) & \text{if } t\le T, \\ 0 & \text{if } t>T, \end{cases} \tag{4.1}$$

where $\psi^{(t)}\in L^1(\mathbb{R}_+,E)$ is defined by

$$\psi^{(t)}(x,u) := \psi(t+x,u), \quad x\in\mathbb{R}_+, \quad u\in\mathbb{R}$$

for $\psi\in L^1(\mathbb{R}_+,E)$. In line with Proposition 3.19, $(\Psi_t^{(t)})_{t\in\mathbb{R}_+}$ may be called *Musiela parametrisation* of the codebook process $(\Psi_t)_{t\in\mathbb{R}_+}$. In other words, the Musiela codebook refers to a function of the remaining life time $x=T-t$ rather than maturity T of the claim.

Function b in (4.1) satisfies Conditions 3–7 in Definition 4.2 if $\check{b}: L^1(\mathbb{R}_+,E)\to L^1(\mathbb{R}_+,E)$ maps $L^1(\mathbb{R}_+,\Pi)$ on a subset of $L^1(\mathbb{R}_+,\Pi)$ and if $\check{b}$ is *locally Lipschitz* in the sense that there exist $x_0,m_0\in\mathbb{R}_+$ such that for any $x\ge x_0, m\ge m_0$ there exists $c\in\mathbb{R}_+$ such that

$$\|\check{b}(\psi_1)-\check{b}(\psi_2)\|_{\widetilde{x},m} \le c\|\psi_1-\psi_2\|_{\widetilde{x},m}$$

holds for any $\psi_1,\psi_2\in L^1(\mathbb{R}_+,E)$ and any $\widetilde{x}\in[x_0,x]$.

4.2. **Existence and uniqueness results.** For building blocks (x_0,ψ_0,b,γ) consider the stochastic differential equation (SDE)

$$d\Psi_t = a(t,\Psi_{t-})dt + b(t,\Psi_{t-})dM_t, \quad \Psi_0=\psi_0 \tag{4.2}$$

in $L^1(\mathbb{R}_+,E)$, where M denotes a subordinator with Lévy exponent $\gamma(0,\cdot)$ and

$$\begin{aligned} a(t,\psi)(T,u) &:= -\partial_T\left(\gamma\Big(u,-i\int_{t\wedge T}^T \tilde{b}(t,\psi)(r,u)dr\Big)\right) \\ &= i\partial_2\gamma\Big(u,-i\int_t^T \tilde{b}(t,\psi)(r,u)dr\Big)\tilde{b}(t,\psi)(T,u)1_{[t,\infty)}(T). \end{aligned} \tag{4.3}$$

with

$$\tilde{b}(t,\psi)(r,u) := \big(\mathrm{Re}(b(t,\psi)(r,u))\wedge 0\big) + i\mathrm{Im}(b(t,\psi)(r,u)). \tag{4.4}$$

Note that $\tilde{b}(t,\psi)=b(t,\psi)$ for $\psi\in\Pi$ by Lemma A.14. In view of Equations (3.10) and (3.1), any compatible codebook process should solve (4.2). We start by showing that (4.2) allows for a unique solution in $L^1(\mathbb{R}_+,E)$.

Proposition 4.6. *Let (x_0,ψ_0,b,γ) be building blocks.* (4.3) *defines a measurable function $a:\mathbb{R}_+\times L^1(\mathbb{R}_+,E)\to L^1(\mathbb{R}_+,E)$. Let $T,T_0,m_0,\widetilde{T},m$ be as in Definition 4.2(5). Then there are a constant $C\in\mathbb{R}_+$ and for any $\|\cdot\|_{\widetilde{T},m}$-bounded set $B\subset L^1(\mathbb{R}_+,E)$ a constant $c\in\mathbb{R}_+$ such that a satisfies the Lipschitz and linear growth conditions*

$$\begin{aligned}\|a(t,\psi_1)-a(t,\psi_2)\|_{\widetilde{T},m} &\le c\|\psi_1-\psi_2\|_{\widetilde{T},m},\\ \|a(t,\psi)\|_{\widetilde{T},m} &\le C(1+\|\psi\|_{\widetilde{T},m})\end{aligned}$$

for any $t\in[0,T]$, $\psi_1,\psi_2\in B$, $\psi\in L^1(\mathbb{R}_+,E)$.
If $(X^\parallel,M)$ denotes an $\mathbb{R}^2$-valued Lévy process with Lévy exponent γ (implying in particular that M is a subordinator), then the SDE (4.2) *has a unique càdlàg $L^1(\mathbb{R}_+,E)$-valued solution, in the sense of $L^1(\mathbb{R}_+,E)$-valued processes and integrals, cf. Section C. The joint law of $(X^\parallel,M,\Psi)$ on*

$$\big(\mathbb{D}(\mathbb{R}^2)\times\mathbb{D}(L^1(\mathbb{R}_+,E)),\mathscr{D}(\mathbb{R}^2)\otimes\mathscr{D}(L^1(\mathbb{R}_+,E))\big)$$

is uniquely determined by (x_0,ψ_0,b,γ). Here, $(\mathbb{D}(\mathbb{R}^2),\mathscr{D}(\mathbb{R}^2))$ and $(\mathbb{D}(L^1(\mathbb{R}_+,E)),\mathscr{D}(L^1(\mathbb{R}_+,E)))$ denote the Skorohod spaces of càdlàg functions on $\mathbb{R}_+$ with values in the Polish spaces $\mathbb{R}^2$ and $L^1(\mathbb{R}_+,E)$, respectively (cf. [17, Section 3.5]*).*

Proof. (1) From the representation

$$a(t,\psi)(T,u)=i\partial_2\gamma\Big(u,-i\int_t^T\tilde{b}(t,\psi)(r,u)dr\Big)\tilde{b}(t,\psi)(T,u)1_{[t,\infty)}(T) \tag{4.5}$$

and boundedness of $\partial_2\gamma$ one concludes that $a(t,\psi)\in L^1(\mathbb{R}_+,E)$. Moreover, a is the composition of the measurable mapping $\mathbb{R}_+\times L^1(\mathbb{R}_+,E)\to\mathbb{R}_+\times L^1(\mathbb{R}_+,E)$, $(t,\psi)\mapsto(t,b(t,\psi))$ and the continuous and hence measurable mapping $\mathbb{R}_+\times L^1(\mathbb{R}_+,E)\to L^1(\mathbb{R}_+,E)$, $(t,\psi)\mapsto f(t,\psi)$ defined by

$$f(t,\psi)(T,u)=i\partial_2\gamma\Big(u,-i\int_t^T\widetilde{\psi}(r,u)dr\Big)\widetilde{\psi}(T,u)1_{[t,\infty)}(T),$$

where $\widetilde{\psi}$ is defined by truncating the real part of ψ as in (4.4).

Let $T,T_0,m_0,\widetilde{T},m,c$ be as in Definition 4.2(5) such that

$$\|b(t,\psi_1)-b(t,\psi_2)\|_{\widetilde{T},m} \le c\|\psi_1-\psi_2\|_{\widetilde{T},m} \tag{4.6}$$

for any $\psi_1,\psi_2\in L^1(\mathbb{R}_+,E)$, $t\in[0,T]$. Let B be a bounded set and L a Lipschitz constant of $\partial_2\gamma$. Then

$$\begin{aligned}H &:= \sup_{(R,u)\in S}\left|i\partial_2\gamma\left(u,-i\int_t^R\widetilde{\psi}_1(r,u)dr\right)-i\partial_2\gamma\left(u,-i\int_t^R\widetilde{\psi}_2(r,u)dr\right)\right|\\ &\le L\sup_{(R,u)\in S}\left|\int_t^R\widetilde{\psi}_1(r,u)dr-\int_t^R\widetilde{\psi}_2(r,u)dr\right|\\ &\le L\sup_{(R,u)\in S}\int_t^R|\psi_1(r,u)-\psi_2(r,u)|dr\\ &\le L\|\psi_1-\psi_2\|_{\widetilde{T},m}\end{aligned}$$

with $S=[t,\widetilde{T}]\times[-m,m]$. Let $c_1\in\mathbb{R}_+$ be a bound for the set B,

$$c_2:=\sup_{t\in[0,T]}\|b(t,0)\|_{\widetilde{T},m}+cc_1,$$

and c_3 be a bound for $\partial_2\gamma$. Observe that $\|b(t,\psi)\|_{\widetilde{T},m} \leq c_2$ for any $\psi \in B, t \in [0,T]$. Submultiplicativity of the uniform norm yields

$$\begin{aligned}
&\|f(t,\psi_1) - f(t,\psi_2)\|_{\widetilde{T},m} \\
&= \int_0^{\widetilde{T}} \sup_{u\in[-m,m]} |(f(t,\psi_1) - f(t,\psi_2))(r,u)|dr \\
&\leq \int_0^{\widetilde{T}} \Big(H \sup_{u\in[-m,m]} |\psi_1(r,u)| \\
&\quad + \sup_{u\in[-m,m]} \left|\partial_2\gamma\left(u, -i\int_t^R \widetilde{\psi}_2(r,u)dr\right)\right| |(\psi_1 - \psi_2)(r,u)|\Big)dr \\
&\leq L\|\psi_1 - \psi_2\|_{\widetilde{T},m}\|\psi_1\|_{\widetilde{T},m} + c_3\|\psi_1 - \psi_2\|_{\widetilde{T},m} \\
&\leq (Lc_2 + c_3)\|\psi_1 - \psi_2\|_{\widetilde{T},m}.
\end{aligned}$$

for any $t \in [0,T]$ and any $\psi_1, \psi_2 \in L^1(\mathbb{R}_+,E)$ which are bounded by c_2. Since a is the composition of $(t,\psi) \to (t,b(t,\psi))$ and f and since b is Lipschitz continuous, the first inequality follows.

For the second inequality note that

$$\begin{aligned}
\|a(t,\psi)\|_{\widetilde{T},m} &\leq c_3\|b(t,\psi)\|_{\widetilde{T},m} \\
&\leq c_3\left(\|b(t,0)\|_{\widetilde{T},m} + \|b(t,\psi) - b(t,0)\|_{\widetilde{T},m}\right) \\
&\leq c_3\left(\|b(t,0)\|_{\widetilde{T},m} + c\|\psi\|_{\widetilde{T},m}\right) \\
&\leq C(1 + \|\psi\|_{\widetilde{T},m})
\end{aligned}$$

by (4.5, 4.6) for $C := c_3(\sup_{t\in[0,T]}\|b(t,0)\|_{\widetilde{T},m} \vee c)$.

(2) For fixed $\omega \in \Omega$ SDE (4.2) is a pathwise equation in the Fréchet space $(L^1(\mathbb{R}_+,E),d)$, driven by two increasing functions. Existence and uniqueness under the present Lipschitz and growth conditions follows from Corollary C.11. Pathwise uniqueness of a solution to SDE (4.2) now implies uniqueness in law. This follows along the same lines as for $\mathbb{R}^d$-valued SDE's driven by a Wiener process, cf. e.g. [33, Theorem IX.1.7 and Exercise IV.5.16]. For the proof of [33, Exercise IV.5.16] one may note that the law of the Bochner integral $\int_0^\cdot a(t,\Psi_{t-})dt$ (and likewise the law of the integral with respect to M) is determined by the law of all random vectors of the form

$$\left(\int_0^{t_1} f_1(a(s,\Psi_{s-}))ds, \dots, \int_0^{t_d} f_d(a(s,\Psi_{s-}))ds\right),$$

where $d \in \mathbb{N}$, $t_1,\dots,t_d \in \mathbb{R}_+$, and $f_1,\dots,f_d$ denote continuous linear functionals on $L^1(\mathbb{R}_+,E)$.

□

We can now state an existence and uniqueness result for compatible option surface models. The condition in Statement 2 of the following theorem means essentially that

- the current codebook state $(T,u) \mapsto \Psi_t(T,u)$ must look like the exponent of a PII whose exponential is a martingale and
- $u \mapsto \Psi_{t-}(t,u)$ is the local exponent of some process X whose exponential is a martingale and whose dependent part $X^{\|}$ relative to M is of the form in Definition 4.2(8).

The first requirement makes sense because of the very idea of a codebook in the present Lévy setup. The second condition, on the other hand, naturally appears through the consistency condition.

Theorem 4.7. *Let (x_0, ψ_0, b, γ) be building blocks.*

(1) *Any two compatible risk-neutral option surface models $(X, \Psi_0, \alpha, \beta, M)$ resp. $(\widetilde{X}, \widetilde{\Psi}_0, \widetilde{\alpha}, \widetilde{\beta}, \widetilde{M})$ coincide in law, i.e. (Ψ, X, M) and $(\widetilde{\Psi}, \widetilde{X}, \widetilde{M})$ have the same law on $\mathbb{D}(L^1(\mathbb{R}_+, E)) \times \mathbb{D}(\mathbb{R}^2)$.*

(2) *If a compatible risk-neutral option surface model $(X, \Psi_0, \alpha, \beta, M)$ exists, then the $L^1(\mathbb{R}_+, E)$-valued process η defined by*
$$\eta_t(T,u) := \Phi_t(T,u) - \gamma(u,0)1_{[0,t]}(T), \quad t, T \in \mathbb{R}_+, \quad u \in \mathbb{R}, \tag{4.7}$$
has values in $L^1(\mathbb{R}_+, \Pi)$. Here, Φ denotes the $L^1(\mathbb{R}_+, E)$-valued solution to SDE (4.2) *from Proposition 4.6.*

(3) *Let Φ denote the $L^1(\mathbb{R}_+, E)$-valued solution to SDE* (4.2) *from Proposition 4.6. If the $L^1(\mathbb{R}_+, E)$-valued process η in (4.7) has values in $L^1(\mathbb{R}_+, \Pi)$, there exists a compatible risk-neutral option surface model.*

Proof. (1) *Step 1:* Let $(X, \Psi_0, \alpha, \beta, M)$ be a risk-neutral option surface model which is compatible with the building blocks. Denote by $X^{\parallel}$ the dependent part of X relative to M. By compatibility we have $\psi^{(X^{\parallel}, M)} = \gamma$. Thus $(X^{\parallel}, M)$ is a Lévy process. Let $(\mathscr{G}_t)_{t \in \mathbb{R}_+}$ be the filtration generated by $(X^{\parallel}, M)$, i.e.
$$\mathscr{G}_t = \bigcap_{s>t} \sigma((X^{\parallel}, M)_r : r \leq s).$$
Since the option surface model $(X, \Psi_0, \alpha, \beta, M)$ is risk neutral, Theorem 3.7 yields that it satisfies the drift condition, the consistency condition and the conditional expectation condition. Moreover, compatibility and the drift condition imply
$$\begin{aligned}
\beta_t(T,u) &= b(t, \Psi_{t-})(T,u), \\
\alpha_t(T,u) &= -\partial_T \left(\gamma\Big(u, -i \int_t^T b(t, \Psi_{t-})(r,u)dr\Big) \right)
\end{aligned}$$
a.s for any $u \in \mathbb{R}$ and almost any $T \in \mathbb{R}_+$, $t \in [0,T]$. (3.1) implies that Ψ solves the SDE
$$d\Psi_t = a(t, \Psi_{t-})dt + b(t, \Psi_{t-})dM_t, \quad \Psi_0 = \psi_0. \tag{4.8}$$
pointwise for any $T \in \mathbb{R}_+$, $u \in \mathbb{R}$. By compatibility, Ψ_t, Ψ_{t-} are $L^1(\mathbb{R}_+, E)$-valued random variables. It is not hard to see that Equation (4.8) holds also in the sense of $L^1(\mathbb{R}_+, E)$-valued processes. By Proposition 4.6 Ψ is the unique pathwise solution to the SDE. Thus Ψ is adapted to the filtration $(\mathscr{G}_t)_{t \in \mathbb{R}_+}$.

Step 2: Define a filtration $(\mathscr{H}_t)_{t \in \mathbb{R}_+}$ via $\mathscr{H}_t := \bigcap_{s>t}(\mathscr{F}_s \vee \mathscr{G}_\infty)$. We show that $X^{\perp} - X_0 = X - X^{\parallel} - X_0$ is a $\mathscr{G}_\infty$-conditional PII with respect to filtration $(\mathscr{H}_t)_{t \in \mathbb{R}_+}$. Indeed, adaptedness follows from the fact that both X and $X^{\perp}$ are adapted to the original filtration $(\mathscr{F}_t)_{t \in \mathbb{R}_+}$. By definition of conditional PII's in [25, Section II.6.1] it remains to be shown that
$$E\left(f(X_r^{\perp} - X_s^{\perp})ZY \right) = E\left(E\big(f(X_r^{\perp} - X_s^{\perp})\big|\mathscr{G}_\infty\big) E(Z|\mathscr{G}_\infty)Y \right) \tag{4.9}$$
for any $s \leq r$, any bounded measurable function $f : \mathbb{R} \to \mathbb{R}$, any bounded $\mathscr{H}_s$-measurable random variable Z and any bounded $\mathscr{G}_\infty$-measurable function Y. By right-continuity of $X^{\perp}$, it suffices to consider only $\mathscr{F}_s \vee \mathscr{G}_\infty$-measurable Z. Standard

measure theory yields that we can focus on functions of the form $f(x)=e^{iux}$ for any $u\in\mathbb{R}$ and Z of the form $Z=1_F1_G$ with $F\in\mathscr{F}_s$ and $G\in\mathscr{G}_\infty$. In view of (4.9), it even suffices to discuss $Z=1_F$ because the second factor can be moved to Y. Moreover, we may replace $X_r^\perp$ by $X^\perp_{r\wedge\tau_n}$, where the $\mathscr{G}_\infty$-measurable stopping times $\tau_n, n\in\mathbb{N}$ are defined by

$$\tau_n := \inf\left\{\tilde{t}\geq s : \mathrm{Re}\Big(\int_s^{\tilde{t}}\Psi_t(t,u)dt\Big)\geq n\right\}.$$

Finally, $\mathscr{G}_\infty$-mesurability of Ψ implies that we can write Y as

$$Y=\widetilde{Y}\exp\left(-\int_s^{r\wedge\tau_n}(\Psi_t(t,u)-\gamma(u,0))dt\right)$$

with some bounded $\mathscr{G}_\infty$-measurable $\tilde{Y}$. The consistency condition and local independence of $X^\parallel, X^\perp$ imply that

$$\Psi_t(t,u)-\gamma(u,0)=\psi_t^X(u)-\psi_t^{X^\parallel}(u)=\psi_t^{X^\perp}(u),\quad u\in\mathbb{R}$$

outside some $dP\otimes dt$-null set. As above, standard measure theory yields that it suffices to consider $\widetilde{Y}$ of the form

$$\widetilde{Y}=\exp\left(i\int_0^T v(t)d(X^\parallel,M)_t\right)$$

with $T\in[r,\infty)$ and bounded measurable $v=(v_1,v_2):[0,T]\to\mathbb{R}^2$. If we set $\mathscr{G}_s^+ := \sigma((X^\parallel,M)_t-(X^\parallel,M)_s : t\geq s)$, we have $\mathscr{G}_\infty=\mathscr{G}_s\vee\mathscr{G}_s^+$. Moreover, $\mathscr{G}_s^+$ is independent of $\mathscr{F}_s$ because $(X^\parallel,M)$ is a Lévy process with respect to filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$. Since $Z=1_F$ is $\mathscr{F}_s$-measurable, we have $E(Z|\mathscr{G}_\infty)=E(Z|\mathscr{G}_s)$, cf. e.g. [3, Satz 54.4]. This yields

$$\begin{aligned} &E\Big(E\big(f(X^\perp_{r\wedge\tau_n}-X^\perp_s)\big|\mathscr{G}_\infty\big)E(Z|\mathscr{G}_\infty)Y\Big)\\ &\quad= E\Big(f(X^\perp_{r\wedge\tau_n}-X^\perp_s)YE(Z|\mathscr{G}_\infty)\Big)\\ &\quad= E\Big(f(X^\perp_{r\wedge\tau_n}-X^\perp_s)YE(Z|\mathscr{G}_s)\Big)\\ &\quad= E\Big(E\big(f(X^\perp_{r\wedge\tau_n}-X^\perp_s)Y\big|\mathscr{G}_s\big)Z\Big). \end{aligned} \tag{4.10}$$

It remains to be shown that $E(f(X^\perp_{r\wedge\tau_n}-X^\perp_s)Y|\mathscr{F}_s)$ is in fact $\mathscr{G}_s$-measurable because together with (4.10) this implies

$$\begin{aligned} E\Big(E\big(f(X^\perp_{r\wedge\tau_n}-X^\perp_s)\big|\mathscr{G}_\infty\big)E(Z|\mathscr{G}_\infty)Y\Big) &= E\Big(E\big(f(X^\perp_{r\wedge\tau_n}-X^\perp_s)Y\big|\mathscr{F}_s\big)Z\Big)\\ &= E\Big(f(X^\perp_{r\wedge\tau_n}-X^\perp_s)ZY\Big) \end{aligned}$$

as claimed in (4.9).

To this end, note that

$$\begin{aligned}
& f(X^\perp_{r\wedge\tau_n}-X^\perp_s)Y \\
&= \exp\left(iu(X^\perp_{r\wedge\tau_n}-X^\perp_s)+i\int_0^T v(t)d(X^\parallel,M)_t-\int_s^{r\wedge\tau_n}\psi_t^{X^\perp}(u)dt\right)\\
&= \exp\Bigg(i\int_0^T (u1_{(s,r\wedge\tau_n]}(t),v_1(t),v_2(t))d(X^\perp,X^\parallel,M)_t \\
&\qquad -\int_0^T \psi_t^{(X^\perp,X^\parallel,M)}(u1_{(s,r\wedge\tau_n]}(t),v_1(t),v_2(t))dt\Bigg)\\
&\qquad \times\exp\left(\int_0^T\psi_t^{(X^\parallel,M)}(v_1(t),v_2(t))dt\right)\\
&= M_TD,
\end{aligned}$$

where $\psi^{(X^\perp,X^\parallel,M)}$, $\psi^{(X^\parallel,M)}$ denote local exponents in the sense of Definition A.4,

$$\begin{aligned}
M &= \exp\Bigg(i\int_0^\cdot (u1_{(s,r\wedge\tau_n]}(t),v_1(t),v_2(t))d(X^\perp,X^\parallel,M)_t \\
&\qquad -\int_0^\cdot \psi_t^{(X^\perp,X^\parallel,M)}(u1_{(s,r\wedge\tau_n]}(t),v_1(t),v_2(t))dt\Bigg),
\end{aligned}$$

and D stands for the remaining factor. Since D is deterministic and M is a bounded local martingale and hence a martingale, we have

$$E(M_TD|\mathscr{F}_s)=M_sD, \tag{4.11}$$

which is $\mathscr{G}_s$-measurable as desired.

Step 3: Using the notation of Step 2, we show that

$$E\big(f(X^\perp_r-X^\perp_s)\big|\mathscr{G}_\infty\big)=\exp\left(\int_s^r(\Psi_t(t,u)-\gamma(u,0))dt\right). \tag{4.12}$$

Indeed, first note that we may replace r with $r\wedge\tau_n$, $n\in\mathbb{N}$ by right-continuity. Choosing $\mathscr{G}_\infty$-measurable Y as in Step 2, we obtain using (4.11):

$$\begin{aligned}
E\big(f(X^\perp_{r\wedge\tau_n}-X^\perp_s)Y\big) &= D\\
&= E(\widetilde{Y})\\
&= E\left(\exp\Big(\int_s^{r\wedge\tau_n}(\Psi_t(t,u)-\gamma(u,0))dt\Big)Y\right),
\end{aligned}$$

which yields the assertion.

Step 4: We now show uniqueness of the law of $(X^\perp,X^\parallel,M,\Psi)$, which implies uniqueness of the law of (Ψ,X,M). To this end, observe that $(X^\parallel,M,\Psi)$ is $\mathscr{G}_\infty$-measurable whereas the conditional law of $X^\perp$ given $\mathscr{G}_\infty$ is determined by the fact that $X^\perp-X_0$ is a $\mathscr{G}_\infty$-conditional PII with conditional characteristic function (4.12). Therefore, it suffices to prove uniqueness of the law of $(X^\parallel,M,\Psi)$. This uniqueness, on the other hand, follows from Step 1 and Statement 2 in Proposition 4.6.

(2) In Step 1 of the proof of Statement 1 it is shown that the codebook process Ψ of (3.1) solves SDE (4.2), i.e. it coincides with Φ. It suffices to show $\int_t^T\eta_s(r,\cdot)dr\in\Pi$ separately for $s<t\le T$ and for $t\le T\le s$.

By definition of option surface models, we have $\int_t^T \Psi_s(r,\cdot)dr \in \Pi$ for $s \leq t \leq T$. Since $\eta_s(r,\cdot) = \Psi_s(r,\cdot)$ for $s < r$, this yields $\int_t^T \eta_s(r,\cdot)dr \in \Pi$ for $s < t \leq T$.

For $r \leq s$ outside some Lebesgue-null set, we have

$$\begin{aligned} \eta_s(r,u) &= \Psi_r(r,u) - \gamma(u,0) \\ &= \psi_r^X(u) - \gamma(u,0) \\ &= \psi_r^{X^\parallel}(u) + \psi_r^{X^\perp}(u) - \gamma(u,0) \\ &= \psi_r^{X^\perp}(u), \quad u \in \mathbb{R}, \end{aligned}$$

where we used the consistency condition in the second equality. By Lemma A.23 and Remark A.17 $u \mapsto \psi_t^{X^\perp}(u)$ is in Π. Lemma 4.1(3) yields that $\int_t^T \eta_s(r,\cdot)dr \in \Pi$ for $t \leq T \leq s$.

(3) *Construction of the codebook process*: By Theorem B.12 there is a Lévy process $(X^\parallel, M)$ on a complete filtered probability space $(\Omega^{(1)}, \mathscr{F}^{(1)}, (\mathscr{F}_t^{(1)})_{t\in\mathbb{R}_+}, P^{(1)})$ such that its (extended) Lévy exponent is γ. Let Ψ be the $L^1(\mathbb{R}_+, E)$-valued càdlàg solution to the SDE

$$d\Psi_t = a(t,\Psi_{t-})dt + b(t,\Psi_{t-})dM_t, \quad \Psi_0 = \psi_0$$

given by Proposition 4.6. Ψ is an $L^1(\mathbb{R}_+,\Pi)$-valued process because this even holds for η by assumption. It is not hard to find versions

$$\begin{aligned} \alpha_t(T,u) &:= a(t,\Psi_{t-})(T,u), \\ \beta_t(T,u) &:= b(t,\Psi_{t-})(T,u) \end{aligned}$$

for any $t,T \in \mathbb{R}_+$, $u \in \mathbb{R}$ and a version of Ψ such that

$$\Psi_t(T,u) = \Psi_0(T,u) + \int_0^{t\wedge T} \alpha_s(T,u)ds + \int_0^{t\wedge T} \beta_s(T,u)dM_s \tag{4.13}$$

for any $t,T \in \mathbb{R}_+$, $u \in \mathbb{R}$ almost surely. More precisely, one can choose versions of the $L^1(\mathbb{R}_+,E)$-valued processes $\Psi, (a(t,\Psi_{t-}))_{t\in\mathbb{R}_+}, (b(t,\Psi_{t-}))_{t\in\mathbb{R}_+}$ such that for any T,u the $\mathbb{C}$-valued process $\Psi(T,u)$ is adapted, almost surely càdlàg, and satisfies (4.13) almost surely.

Construction of the return process: As usual, let $(\mathbb{D},\mathscr{D},(\mathscr{D}_t)_{t\in\mathbb{R}_+})$ denote the Skorokhod space of real-valued càdlàg functions. Let $X^\perp$ be the canonical process on $\mathbb{D}$ and set

$$\begin{aligned} \Omega &:= \Omega^{(1)} \times \mathbb{D}, \\ \mathscr{F} &:= \mathscr{F}^{(1)} \otimes \mathscr{D}, \\ \mathscr{F}_t &:= \bigcap_{s>t} \left(\mathscr{F}_s^{(1)} \otimes \mathscr{D}_s\right). \end{aligned}$$

Fix $\omega_1 \in \Omega^{(1)}$. Theorem B.12 yields that there is a probability measure $P^{(2)}(\omega_1,\cdot)$ on $(\mathbb{D},\mathscr{D})$ such that $X_0^\perp = x_0$ a.s. and $X^\perp - X_0^\perp$ is a PII with characteristic function

$$u \mapsto \exp\left(\int_0^t \eta_\infty(r,u)(\omega_1)dr\right) = \exp\left(\int_0^t \eta_t(r,u)(\omega_1)dr\right). \tag{4.14}$$

Measurability of η implies that $P^{(2)}$ is a transition kernel from $(\Omega^{(1)},\mathscr{F}^{(1)})$ to $(\mathbb{D},\mathscr{D})$. Therefore,

$$P(d(\omega_1,\omega_2)) := (P^{(1)} \otimes P^{(2)})(d(\omega_1,\omega_2)) := P^{(1)}(d\omega_1)P^{(2)}(\omega_1,d\omega_2)$$

defines a probability measure P on $(\Omega,\mathscr{F})$.

By abuse of notation we will use the same letters for the process $M,\Psi,X^{\parallel},X^{\perp}$ embedded in the filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in\mathbb{R}_+},P)$, i.e. we denote e.g. the process $((\omega_1,\omega_2),t)\mapsto M_t(\omega_1)$ again by M. Set

$$X := X^{\parallel}+X^{\perp} \tag{4.15}$$

Observe that $X^{\perp}-X_0$ is an $\mathscr{F}^{(1)}\otimes\{\emptyset,\mathbb{D}\}$-conditional PII relative to the filtration $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$ defined by

$$\mathscr{G}_t := \bigcap_{s>t}\big(\mathscr{F}^{(1)}\otimes\mathscr{D}_s\big).$$

Denote by (b,c,K) its local characteristics relative to some truncation function h. Then [25, Theorem II.6.6] yields

$$iub_t(\omega)-\frac{u^2}{2}c_t(\omega)+\int(e^{iux}-1-iuh(x))K_t(\omega,dx)=\eta(t,u)(\omega_1)$$

for almost all $\omega=(\omega_1,\omega_2)\in\Omega$, $t\in\mathbb{R}_+$, $u\in\mathbb{R}$. Both $X^{\perp}$ and (b,c,K) are $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-adapted. From Proposition A.21 it follows that $X^{\perp}$ is a semimartingale with respect to this smaller filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ with the same local characteristics (b,c,K).

We want to show that $(X^{\parallel},M)$ is a Lévy process on $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in\mathbb{R}_+},P)$ as well. By right-continuity of $(X^{\parallel},M)$ it suffices to prove that

$$E\big(U\big|\mathscr{F}_s^{(1)}\otimes\mathscr{D}_s\big)=E(U)$$

for any $s,t\in\mathbb{R}_+$ with $s\le t$ and any $u\in\mathbb{R}^2$, where

$$U:=\exp\Big(iu\big((X^{\parallel},M)_t-(X^{\parallel},M)_s\big)\Big).$$

It suffices to show that

$$E(UVW)=E(U)E(VW)$$

for any bounded $\mathscr{F}_s^{(1)}\otimes\{\emptyset,\mathbb{D}\}$-measurable V and any bounded $\{\emptyset,\Omega^{(1)}\}\otimes\mathscr{D}_s$-measurable W. The conditional law of $(X_r^{\perp})_{r\le s}$ given $\mathscr{F}^{(1)}\otimes\{\emptyset,\mathbb{D}\}$ is $\mathscr{F}_s^{(1)}\otimes\{\emptyset,\mathbb{D}\}$-measurable because $\eta(r,u)$ is $\mathscr{F}_s^{(1)}$-measurable for any $r\le s$. This implies

$$E\big(W\big|\mathscr{F}^{(1)}\otimes\{\emptyset,\mathbb{D}\}\big)=E\big(W\big|\mathscr{F}_s^{(1)}\otimes\{\emptyset,\mathbb{D}\}\big)$$

because W is a measurable function of $(X_r^{\perp})_{r\le s}$. Moreover, U is independent of $\mathscr{F}_s^{(1)}\otimes\{\emptyset,\mathbb{D}\}$ because $(X^{\parallel},M)$ is a Lévy process on $\Omega^{(1)}$. This yields

$$\begin{aligned}
E(UVW) &= E\Big(UVE\big(W\big|\mathscr{F}^{(1)}\otimes\{\emptyset,\mathbb{D}\}\big)\Big)\\
&= E\Big(UVE\big(W\big|\mathscr{F}_s^{(1)}\otimes\{\emptyset,\mathbb{D}\}\big)\Big)\\
&= E\Big(E\big(U\big|\mathscr{F}_s^{(1)}\otimes\{\emptyset,\mathbb{D}\}\big)VW\Big)\\
&= E(U)E(VW)
\end{aligned}$$

as desired.

Compatibility of the constructed model: We have $\beta_t=b(t,\Psi_{t-})$, $X_0=x_0$, $\Psi_0=\psi_0$ and $\gamma=\psi^{(X^{\parallel},M)}$. We must show that the dependent part of X relative to M is $X^{\parallel}$. Since $X^{\parallel}$ is the dependent part of $X^{\parallel}$ relative to M, it remains to be shown that M and $X^{\perp}$ are locally independent. Since M is a subordinator, it suffices to prove that

$$P(\exists t\in\mathbb{R}_+:\Delta M_t\neq 0,\Delta X_t^{\perp}\neq 0)=0.$$

Let $J := \{t \in \mathbb{R}_+ : \Delta M_t \neq 0\}$ denote the set of jump times of M. Then J is almost surely countable and

$$\begin{aligned} P\big(\exists t \in \mathbb{R}_+ : \Delta M_t \neq 0, \Delta X_t^\perp \neq 0\big) &\leq E\left(\sum_{s\in J} 1_{\{\Delta X_s^\perp \neq 0\}}\right) \\ &= E\left(\sum_{s\in J} P\big(\Delta X_s^\perp \neq 0 \big| M\big)\right) \\ &= 0 \end{aligned}$$

because

$$E\big(\exp(iu\Delta X_s^\perp)\big|\mathscr{F}^{(1)} \otimes \{\emptyset, \mathbb{D}\}\big) = \exp\left(\int_{s-}^{s} \eta(r,u)dr\right) = 0, \quad u \in \mathbb{R}$$

and hence $P(\Delta X_s^\perp \neq 0|M) = 0$.

Risk neutrality of the constructed model: The constructed model satisfies the consistency and the drift condition. Hence Theorem 3.7 yields risk-neutrality.

□

Examples illustrating the previous result are to be found in Sections 4.3 and 4.4 below. In general, however, it is not obvious why the solution to SDE (4.2) should satisfy the condition in Statement 3 of Theorem 4.7. If this is not the case, a compatible risk-neutral option surface model does not exist. As a way out, we introduce a weaker form of compatibility, which assumes (4.2) to hold only up to some maximal stopping time. For related discussions on stochastic invariance problems, we refer the reader to [19, 20].

Definition 4.8. (1) An option surface model $(X, \Psi_0, \alpha, \beta, M)$ is called τ-*weakly compatible* with building blocks (x_0, ψ_0, b, γ) if
- τ is a stopping time,
- $X_0 = x_0$,
- $\Psi_0 = \psi_0$,
- $\mathbb{R}_+ \to L^1(\mathbb{R}_+, E)$, $t \mapsto \Psi_t(\omega)$ is well defined and a.s. càdlàg,
- $\beta_t(\omega) = b(t, \Psi_{t-}(\omega))$ for $dP \otimes dt$-almost any $(\omega, t) \in [\![0, \tau]\!]$,
- $\psi^{(X^{\Vdash}, M)}(u,v) = \gamma(u,v)$ for $(u,v) \in \mathbb{R} \times (\mathbb{R} + i\mathbb{R}_+)$, where $X^{\Vdash}$ denotes some process which coincides on $[\![0, \tau]\!]$ with the dependent part $X^{\parallel}$ of X relative to M.

(2) Let $(X, \Psi_0, \alpha, \beta, M)$ denote an option surface model which is τ-weakly compatible with building blocks (x_0, ψ_0, b, γ). It is called *maximal weakly compatible* if
- $$\tau = \inf\big\{t \in \mathbb{R}_+ : \eta_t \notin L^1(\mathbb{R}_+, \Pi)\big\} \quad \text{a.s,} \tag{4.16}$$
  where Φ denotes the unique $L^1(\mathbb{R}_+, E)$-valued solution to SDE (4.2) from Proposition 4.6 and η is defined as in (4.7),
- $t \mapsto \Psi_t(T, u)$ from (3.1) stays constant after τ.

We can now state our general existence and uniqueness result.

Theorem 4.9. *Let (x_0, ψ_0, b, γ) be building blocks.*

(1) *There exists a maximal weakly compatible and risk-neutral option surface model.*

(2) *Any two maximal weakly compatible and risk-neutral option surface models $(X, \Psi_0, \alpha, \beta, M)$, $(\widetilde{X}, \widetilde{\Psi}_0, \widetilde{\alpha}, \widetilde{\beta}, \widetilde{M})$ coincide in law, i.e. (Ψ, X, M) and $(\widetilde{\Psi}, \widetilde{X}, \widetilde{M})$ have the same law on $\mathbb{D}(L^1(\mathbb{R}_+, E)) \times \mathbb{D}(\mathbb{R}^2)$.*

(3) *If a compatible risk-neutral option surface model exists, then any maximal weakly compatible risk-neutral option surface model is in fact compatible.*

Proof. (1) Let $(X^{\Vdash}, M)$ be a Lévy process with characteristic exponent γ. Define a as in Proposition 4.6, Φ as the unique $L^1(\mathbb{R}_+, E)$-valued solution to SDE (4.2), and the $L^1(\mathbb{R}_+, E)$-valued adapted càdlàg process η as in (4.7). [17, Problem 2.9.1, Theorem 2.1.6] and [25, Lemma I.1.19] yield that there is a stopping time τ which satisfies Equation (4.16). Set

$$\begin{aligned}
\Psi_t &:= \Phi_{t\wedge\tau}, \\
\alpha_t(T,u) &:= a(t,\Psi_{t-})(T,u)1_{[\![0,\tau]\!]}(t), \qquad (4.17)\\
\beta_t(T,u) &:= b(t,\Psi_{t-})(T,u)1_{[\![0,\tau]\!]}(t). \qquad (4.18)
\end{aligned}$$

More specifically, it is not hard to find versions of the right-hand sides of (4.17,4.18) such that (4.13) holds up tp τ. We have $\Psi_0 = \psi_0$. Along the same lines as in the proof of Theorem 4.7 we can now construct a return process X such that $(X, \Psi_0, \alpha, \beta, M)$ is a τ-weakly compatible and risk-neutral option surface model. More specifically, η in (4.14) must be replaced by $\eta_t^\tau(\cdot,\cdot) := \eta_{t\wedge\tau}(\cdot,\cdot)$ and X in (4.15) by $X := (X^{\Vdash})^\tau + X^\perp$. The option surface model $(X, \Psi_0, \alpha, \beta, M)$ is in fact maximal weakly compatible.

(2) By Proposition 4.6(2), the law of $(X^{\Vdash}, M, \Phi)$ is uniquely determined by (x_0, ψ_0, b, γ), where Φ denotes the solution to SDE (4.2). [17, Problem 2.9.1, Theorem 2.1.6] and [25, Lemma I.1.19] yield that there is a stopping time τ which satisfies Equation (4.16). Consider now the following stopped variant of SDE (4.2):

$$d\Psi_t = a(t,\Psi_{t-})1_{[\![0,\tau]\!]}(t)dt + b(t,\Psi_{t-})1_{[\![0,\tau]\!]}(t)dM_t, \quad \Psi_0 = \psi_0. \qquad (4.19)$$

The uniqueness statement follows now along the same lines as in the proof of Theorem 4.7 if we replace (4.2) by (4.19), $X^{\|}$ by $(X^{\Vdash})^\tau$, and $\gamma(u,0)$ by $\gamma(u,0)1_{[\![0,\tau]\!]}$.

(3) Let $(X, \Psi_0, \alpha, \beta, M)$ be a maximal weakly compatible and risk-neutral option surface model, Φ the unique $L^1(\mathbb{R}_+, E)$-valued solution to SDE (4.2), and τ as in (4.16). Theorem 4.7(2) yields $\tau = \infty$ because there exists some compatible and risk-neutral option surface model. This implies that $(X, \Psi_0, \alpha, \beta, M)$ is a compatible option surface model.

□

4.3. **Vanishing coefficient process β.** The simplest conceivable codebook model (2.13) is obtained for building blocks (x_0, ψ_0, b, γ) where $b = 0$ or equivalently $\gamma = 0$. Not surprisingly, it leads to constant codebook processes and hence to the simple model class that we used to motivate option surface models in Section 2.2.

Corollary 4.10. *Let (x_0, ψ_0, b, γ) be building blocks with $\gamma = 0$. Then there is a compatible risk-neutral option surface model $(X, \Psi_0, \alpha, \beta, M)$. For any such model, $X - X_0$ is a PII with characteristic function*

$$E\big(\exp(iu(X_T - X_0))\big) = \exp\left(\int_0^T \psi_0(r,u)dr\right), \quad u \in \mathbb{R}, \quad T \in \mathbb{R}_+.$$

In particular, the law of X is uniquely determined.

Proof. Let Ψ be the solution to SDE (4.2) given by Proposition 4.6. Then we have $\Psi_t(T,u) = \psi_0(T,u)$ for all $t, T \in \mathbb{R}_+, u \in \mathbb{R}$ because $M = 0$ and $\alpha = 0$. In particular, Ψ is an $L^1(\mathbb{R}_+, \Pi)$-valued process. Thus the existence of a compatible risk-neutral option surface model

$(X,\Psi_0,\alpha,\beta,M)$ follows from Theorem 4.7. Theorem 3.7 yields that $(X,\Psi_0,\alpha,\beta,M)$ satisfies the conditional expectation condition. Hence

$$E\left(e^{iu(X_T-X_t)}\middle|\mathscr{F}_t\right)=\exp\left(\int_t^T\psi_0(r,u)dr\right).$$

In particular, $X-X_0$ is a PII by definition. □

The Black-Scholes model is obtained for a particular choice of the initial state of the codebook.

Example 4.11 (Black-Scholes model)**.** If we choose $\psi_0(T,u):=-(iu+u^2)\sigma^2/2$ in Theorem 4.10 for some $\sigma>0$, we obtain $E(e^{iuX_T})=\exp(iuX_0-iu\frac{\sigma^2}{2}T-\frac{u^2}{2}\sigma^2T)$, which means $X_T\sim N(X_0-\frac{\sigma^2}{2}T,\sigma^2T)$, $T\in\mathbb{R}_+$. Put differently, the return process X is Brownian motion with drift rate $-\sigma^2/2$ and volatility σ.

4.4. **Deterministic coefficient process β.** In this section we consider building blocks (x_0,ψ_0,b,γ) where b depends on the time parameter only. Then a defined as in Equation (4.3) also depends only on time. Thus SDE (4.2) is solved by mere integration. In the following, we omit the redundant argument ψ and write $b(t)$, $a(t)$ for $b(t,\psi)$, $a(t,\psi)$, respectively.

Corollary 4.12. *Let (x_0,ψ_0,b,γ) be building blocks such that b is constant in its second variable, i.e. $\beta_t(T,u):=b(t,\psi)(T,u)$ does not depend on ψ.*

(1) *Then there exists a maximal weakly compatible risk-neutral option surface model. Any two such models $(X,\Psi_0,\alpha,\beta,M),(\widetilde{X},\widetilde{\Psi}_0,\widetilde{\alpha},\widetilde{\beta},\widetilde{M})$ coincide in law, i.e. (Ψ,X,M) and $(\widetilde{\Psi},\widetilde{X},\widetilde{M})$ have the same law on $\mathbb{D}(L^1(\mathbb{R}_+,E))\times\mathbb{D}(\mathbb{R}^2)$.*
(2) *Let a be defined as in* (4.3)*. The model in Statement 1 is compatible if and only if the deterministic mappings*

$$\varphi_t:(T,u)\mapsto\psi_0(T,u)+\int_0^t a(s)(T,u)ds-\gamma(u,0)1_{[0,t]}(T)\tag{4.20}$$

are in $L^1(\mathbb{R}_+,\Pi)$ for any $t\in\mathbb{R}_+$.

Proof. (1) The first assertion follows from Theorem 4.9.

(2) $\Rightarrow$: Suppose that a compatible risk-neutral option surface model exists. Fix $t\in\mathbb{R}_+$. Since the mapping φ_t in (4.20) is in $L^1(\mathbb{R}_+,E)$, it remains to be shown that $\int_{T_1}^{T_2}\varphi_t(r,\cdot)dr\in\Pi$ for any $t\in\mathbb{R}_+$ and any $T_1\le T_2$. For η_t as in (4.7) we have that

$$\begin{aligned}\int_{T_1}^{T_2}\eta_t(r,\cdot)dr-\int_{T_1}^{T_2}\varphi_t(r,\cdot)dr &= \int_{T_1}^{T_2}\int_0^{t\wedge T_2}b(s)(r,\cdot)dM_sdr\\ &= \int_0^{t\wedge T_2}\int_{T_1}^{T_2}\beta_s(r,\cdot)drdM_s\end{aligned}\tag{4.21}$$

is in Π by Lemma 4.1(5). Corollary B.9 yields

$$\left|\int_0^{t\wedge T_2}\int_{T_1}^{T_2}\beta_s(r,u)drdM_s\right|(\omega_n)\underset{n\to\infty}{\longrightarrow}0$$

for some sequence $(\omega_n)_{n\in\mathbb{N}}$ in Ω. The proof actually shows that the same sequence can be chosen for all $u\in\mathbb{R}$. Hence there is a sequence $(\omega_n)_{n\in\mathbb{N}}$ in Ω such that

$$\left(\int_{T_1}^{T_2}\eta_t(r,\cdot)dr\right)(\omega_n)\in\Pi$$

for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty}\left(\int_{T_1}^{T_2}\eta_t(r,u)dr\right)(\omega_n) = \int_{T_1}^{T_2}\varphi_t(r,u)dr$$

for all $u \in \mathbb{R}$. Since the continuous mapping $u \mapsto \int_{T_1}^{T_2}\varphi_t(r,u)dr$ is the pointwise limit of characteristic exponents of infinitely divisible distributions, [30, Theorem 5.3.3] yields that it is a characteristic exponent of an infinitely divisible distribution as well. By (4.21) it is a difference of functions in Π and hence itself in Π.

$\Leftarrow$: Suppose conversely that the mappings φ_t in (4.20) are in $L^1(\mathbb{R}_+,\Pi)$. Since

$$\begin{aligned}\eta_t(T,u) &= \Phi_t(T,u) - \gamma(u,0)1_{[0,t]}(T)\\ &= \varphi_t(T,u) + \int_0^t b(t)(T,u)dM_t,\end{aligned}$$

the assumption and Lemma 4.1(5) yield $\eta_t \in L^1(\mathbb{R}_+,\Pi)$.

□

The condition in Statement 2 of Corollary 4.12 means that the initial codebook state ψ_0 must be greater or equal than

$$\mu_t : (T,u) \mapsto \gamma(u,0)1_{[0,t]}(T) - \int_0^t a(s)(T,u)ds$$

in the sense that $\psi_0 - \mu_t \in L^1(\mathbb{R}_+,\Pi)$ for any $t \in \mathbb{R}_+$. Put differently, the initial option prices must be large enough to allow for a compatible risk-neutral option surface model.

Remark 4.13. For deterministic β as in Corollaries 4.10 and 4.12 it may not be obvious why one should require the càdlàg property of the codebook in Definition 4.4. However, in this case it holds automatically. Indeed, let (x_0,ψ_0,b,γ) be building blocks and $(X,\Psi_0,\alpha,\beta,M)$ a compatible risk-neutral option surface model. Then α and β are deterministic and

$$\Psi_t(T,u) = \Psi_0(T,u) + \int_0^t \alpha_s(T,u)ds + \int_0^t \beta_s(T,u)dM_s.$$

It is not hard to conclude that

$$\Psi_t = \Psi_0 + \int_0^t \alpha_s ds + \int_0^t \beta_s dM_s$$

holds in the sense of $L^1(\mathbb{R}_+,E)$-valued integrals as well. Since the right-hand side is càdlàg, $t \mapsto \Psi_t$ is càdlàg as well.

If b in Corollary 4.12 is constant in Musiela parametrisation, i.e. if it is of the form (4.1) with a constant $\check{b}$, the compatibility condition in Statement 2 of this Corollary can be simplified.

Lemma 4.14. *Assume that b is of the form*

$$b(t)(T,u) = \check{b}(T-t,u)1_{[t,\infty)}(T), \quad t,T \in \mathbb{R}_+, u \in \mathbb{R}$$

for some $\check{b} \in L^1(\mathbb{R}_+,\Pi)$ *and let a be defined by Equation* (4.3) *as usual. Then*

$$(T,u) \mapsto -\int_0^T a(s)(T,u)ds + \gamma(u,0) \tag{4.22}$$

is in $L^1(\mathbb{R}_+,\Pi)$. *If the mapping* $\varphi_\infty \in L^1(\mathbb{R}_+,E)$ *defined by*

$$\varphi_\infty(T,u) := \psi_0(T,u) + \int_0^T a(s)(T,u)ds - \gamma(u,0)$$

is in $L^1(\mathbb{R}_+,\Pi)$ *as well, then* φ_t *from (4.20) is in* $L^1(\mathbb{R}_+,\Pi)$ *for any* $t\in\mathbb{R}_+$.

Proof. First note that $\varphi_t \in L^1(\mathbb{R}_+,E)$ for any $t\in\overline{\mathbb{R}}_+$. Since

$$\begin{aligned} a(t)(T,u) &= -\partial_T\left(\gamma\left(u,-i\int_{t\wedge T}^{T}\check{b}(r-t,u)dr\right)\right)\\ &= -\partial_T\left(\gamma\left(u,-i\int_{(T-t)\wedge 0}^{T-t}\check{b}(r,u)dr\right)\right)\\ &= \partial_t\left(\gamma\left(u,-i\int_{(T-t)\wedge 0}^{T-t}\check{b}(r,u)dr\right)\right)\end{aligned}$$

we have

$$\int_0^T a(s)(T,u)ds = \gamma(u,0)-\gamma\left(u,-i\int_0^T\check{b}(r,u)dr\right) \tag{4.23}$$

and

$$\begin{aligned}\int_0^t a(s)(T,u)ds &= \gamma\left(u,-i\int_{(T-t)\wedge 0}^{T-t}\check{b}(r,u)dr\right)-\gamma\left(u,-i\int_0^T\check{b}(r,u)dr\right)\\ &= \gamma\left(u,-i\int_{(T-t)\wedge 0}^{T-t}\check{b}(r,u)dr\right)+\int_0^T a(s)(T,u)ds-\gamma(u,0)\end{aligned}$$

for any $t,T\in\mathbb{R}_+$, $u\in\mathbb{R}$. Consequently

$$\varphi_t(T,u)=\varphi_\infty(T,u)+\gamma\left(u,-i\int_0^{T-t}\check{b}(r,u)dr\right)1_{[t,\infty)}(T) \tag{4.24}$$

for any $t,T\in\mathbb{R}_+$, $u\in\mathbb{R}$. [34, Theorem 30.1] yields that

$$u\mapsto\gamma\left(u,-i\int_0^{T-t}\check{b}(r,u)dr\right)1_{[t,\infty)}(T)$$

is in Π and hence the second summand of φ_t in (4.24) is in $L^1(\mathbb{R}_+,\Pi)$ by Lemma 4.1(4). Since this holds for the first summand as well, the second statement follows. Similarly, we have that the last term in (4.23) is in $L^1(\mathbb{R}_+,\Pi)$, which yields the first statement. □

Since (4.22) is in $L^1(\mathbb{R}_+,\Pi)$, the condition in Lemma 4.14 means that the initial codebook ψ_0 must be the sum of this minimal codebook (4.22) and any other element of $L^1(\mathbb{R}_+,\Pi)$.

If β is of product form, we can establish a link to affine Markov processes in the sense of [18].

Theorem 4.15. *Let* (x_0,ψ_0,b,γ) *be building blocks such that*

(1)
$$b(t)(T,u)=\varphi(u)\exp\left(-\int_t^T\lambda(s)ds\right)1_{[t,\infty)}(T)$$
for some $\varphi\in\Pi$ *and some continuous function* $\lambda:\mathbb{R}_+\to\mathbb{R}$,

(2) $\psi_0(T,u)$ *is continuous in* T *for fixed* u,

(3) *the compatibility condition in Statement 2 of Corollary 4.12 holds.*

Then (X,Z) *is a time-inhomogeneous affine process in the sense of* [18], *where* $(X,\Psi_0,\alpha,\beta,M)$ *denotes a compatible risk-neutral option surface model (cf. Corollary 4.12) and*

$$Z_t:=\int_0^t\exp\left(-\int_s^t\lambda(s)ds\right)dM_s.$$

Proof. Let $X^{\parallel}$ be the dependent part of X relative to M and $X^{\perp} := X - X^{\parallel}$. The local exponent of $X^{\perp}$ satisfies

$$\begin{aligned}
\psi_t^{X^{\perp}}(u) &= \psi_t^X(u) - \psi^{X^{\parallel}}(u) \\
&= \Psi_0(t,u) + \int_0^t \alpha_s(t,u)ds + \int_0^{t-} \beta_s(t,u)dM_s - \psi^{X^{\parallel}}(u) \\
&= \Psi_0(t,u) + \int_0^t \alpha_s(t,u)ds + \varphi(u)Z_{t-} - \psi^{X^{\parallel}}(u) \\
&= \Psi_0(t,u) - \int_0^t \partial_t\left(\psi^{(X^{\parallel},M)}\left(u, -i\int_s^t \beta_s(r,u)dr\right)\right)ds \\
&\quad + \varphi(u)Z_{t-} - \psi^{X^{\parallel}}(u),
\end{aligned}$$

where we used the consistency condition in the second and the drift condition (3.10) in the last equality. Since

$$dZ_t = -\lambda(t)Z_t dt + dM_t$$

and by Lemma A.23, we obtain for the local exponent of (X,M,Z):

$$\begin{aligned}
\psi_t^{(X,M,Z)}(u,v,w) &= \psi_t^{(X,M)}(u,v+w) - iw\lambda(t)Z_{t-} \\
&= \psi^{(X^{\parallel},M)}(u,v+w) - iw\lambda(t)Z_{t-} + \psi_t^{X^{\perp}}(u) \\
&= \Phi_0(t;u,v,w) + \Phi_1(t;u,v,w)Z_{t-}
\end{aligned}$$

with

$$\begin{aligned}
\Phi_0(t;u,v,w) &:= \psi^{(X^{\parallel},M)}(u,v+w) - \psi^{X^{\parallel}}(u) \\
&\quad + \Psi_0(t,u) - \int_0^t \partial_t\left(\psi^{(X^{\parallel},M)}\left(u, -i\int_s^t \beta_s(r,u)dr\right)\right)ds, \\
\Phi_1(t;u,v,w) &:= \varphi(u) - iw\lambda(t).
\end{aligned}$$

This implies that $(u,v,w) \mapsto \Phi_0(t;u,v,w) + \Phi_1(t;u,v,w)Z_{t-}$ is a Lévy exponent on $\mathbb{R}^3$. Since $\operatorname{ess\,inf} M_t = 0$ we have $\operatorname{ess\,inf} Z_t = 0$ by Corollary B.9. At the end of this proof we show that $(u,v,w) \mapsto \Phi_0(t;u,v,w)$ is a Lévy exponent for fixed t. The same holds for Φ_1. Relative to some truncation function h, denote by $(\beta_t^{(0)}, \gamma_t^{(0)}, \kappa_t^{(0)})$, $(\beta_t^{(1)}, \gamma_t^{(1)}, \kappa_t^{(1)})$ Lévy-Khintchine triplets on $\mathbb{R}^3$ which correspond to $\Phi_0(t;\cdot)$ and $\Phi_1(t;\cdot)$ respectively. Observe that $\Phi_0(t;u,v,w)$ and $\Phi_1(t;u,v,w)$ are continuous in t for fixed (u,v,w). Lévy's continuity theorem and [25, Theorem VII.2.9] imply that $(\beta_t^{(0)}, \gamma_t^{(0)}, \kappa_t^{(0)})$, $(\beta_t^{(1)}, \gamma_t^{(1)}, \kappa_t^{(1)})$ are continuous in t in the sense of Conditions $[\beta_1], [\gamma_1], [\delta_{1,3}]$ in that theorem. A detailed inspection of the arguments shows that this weaker continuity suffices for the proof of [18, Proposition 4.1]. The assertion follows now from [18, Theorem 2.14].

Let $t \in \mathbb{R}_+$. Since $\operatorname{ess\,inf} Z_t = 0$ there is a sequence $\omega_n \in \Omega$ such that $\psi_n := \psi_t^{(X,M,Z)}(\omega_n)$ is a Lévy exponent and $Z_{t-}(\omega_n) \to 0$ for $n \to \infty$. Then $\psi_n \to \Phi_0(t;\cdot)$ locally uniformly. Thus [34, Proposition 2.5 and Lemma 7.8] yield that $\Phi_0(t;\cdot)$ is a Lévy exponent. □

Finally, we consider more specific choices of φ, λ.

Corollary 4.16. *Let (x_0, ψ_0, b, γ) be building blocks such that*

$$\begin{aligned}
b(t,\psi)(T,u) &= b(t)(T,u) := \varphi(u)e^{-\lambda(T-t)}1_{[t,\infty)}(T), \\
\gamma(u,v) &= \eta(\delta u + v) - iu\eta(-\delta i)
\end{aligned}$$

for $t,T\in\mathbb{R}_+,\psi\in L^1(\mathbb{R}_+,\Pi),u\in\mathbb{R},v\in\mathbb{R}+i\mathbb{R}_+$, *where* $\lambda\in(0,\infty)$, $\delta\in\mathbb{R}_-$, $\varphi(u):=-(u^2+iu)/2$, $u\in\mathbb{C}$, *and* $\eta:\mathbb{R}+i\mathbb{R}_+\to\mathbb{C}$ *denotes the extended Lévy exponent of a pure-jump subordinator with finite second moments. Suppose that*

$$\psi_0(T,u)=\psi^L(T,u)+\eta\left(\delta u-i\varphi(u)\frac{1-e^{-\lambda T}}{\lambda}\right)-iu\eta(-\delta i)$$

for some $\psi^L\in L^1(\mathbb{R}_+,\Pi)$ *(which implies* $\psi_0\in L^1(\mathbb{R}_+,\Pi)$ *because it is the sum of two objects in* $L^1(\mathbb{R}_+,\Pi)$*).*

Then there is a compatible risk-neutral option surface model $(X,\Psi_0,\alpha,\beta,M)$*. Moreover, it can be chosen such that there is a standard Wiener process* W *and a time-inhomogeneous Lévy process* L *with characteristic function*

$$E(e^{iuL_T})=\exp\left(\int_0^T\psi^L(r,u)dr\right),\quad T\in\mathbb{R}_+,u\in\mathbb{R},$$

W,L,M *are independent, and*

$$\begin{aligned} dX_t &= dL_t-\left(\frac{1}{2}Z_t+\eta(-\delta i)\right)dt+\sqrt{Z_t}dW_t+\delta dM_t, && (4.25)\\ dZ_t &= -\lambda Z_t dt+dM_t && (4.26)\end{aligned}$$

holds with $X_0=x_0$, $Z_0=0$.

Proof. Step 1: Let Y be a Lévy process with Lévy exponent

$$\psi:\mathbb{C}\to\mathbb{C},\quad u\mapsto i\delta u+\varphi(u)\frac{1-e^{-\lambda T}}{\lambda}$$

and M an independent subordinator with exponent η. Set $U_t:=Y_{M_t}-t\eta(-i\delta)$. Observe that $\eta(-i\delta)\in\mathbb{R}_-$. Then [34, Theorem 30.1] yields that U is a Lévy process with Lévy exponent

$$u\mapsto\eta\left(\delta u-i\varphi(u)\frac{1-e^{-\lambda T}}{\lambda}\right)-iu\eta(-\delta i).$$

Moreover, [34, Theorem 30.1] also implies that

$$P(U_t\in B)=\int P(Y_s-t\eta(-i\delta)\in B)P^{M_t}(ds)$$

for any $B\in\mathscr{B}$. Thus

$$\begin{aligned} E(e^{U_1}) &= e^{-\eta(-i\delta)}\int E(e^{Y_s})P^{M_1}(ds)\\ &= e^{-\eta(-i\delta)}\int\exp(\psi(-i)s)P^{M_1}(ds)\\ &= e^{-\eta(-i\delta)}\int\exp(\delta s)P^{M_1}(ds)\\ &= e^{-\eta(-i\delta)}\exp(\eta(-i\delta))\\ &= 1,\end{aligned}$$

which implies that the Lévy exponent of U is an element of Π. Lemma 4.1(4) yields that

$$(T,u)\mapsto\eta\left(\delta u-i\varphi(u)\frac{1-e^{-\lambda T}}{\lambda}\right)-iu\eta(-\delta i)$$

is an element of $L^1(\mathbb{R}_+,\Pi)$.

Step 2: Let W,L,M be independent Lévy processes on a filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in\mathbb{R}_+},P)$ such that W is a Brownian motion, L is a PII with characteristic function

$$E(e^{iuL_T}) = \exp\left(\int_0^T \psi^L(r,u)dr\right), \quad T\in\mathbb{R}_+, u\in\mathbb{R},$$

and M a Lévy process with characteristic exponent η. Let (X,Z) be a solution to the system of SDE's (4.25, 4.26). The dependent part of X relative to M is

$$X^{\|} := (\delta M_t - \eta(-\delta i)t)_{t\in\mathbb{R}_+}$$

because W,L are independent of M and $\psi^{X^{\|}}(-i) = \eta(-\delta i) - \eta(-\delta i) = 0$. Moreover, $(X^{\|},M)$ has Lévy exponent γ. Define

$$\begin{aligned}\Psi_t(T,u) &:= \psi_0(T,u) + \int_0^t a(s)(T,u)ds + \varphi(u)e^{-\lambda(T-(t\wedge T))}Z_{t\wedge T}\\ &= \psi_0(T,u) + \int_0^t a(s)(T,u)ds + \int_0^t b(s)(T,u)dM_s\end{aligned}$$

with

$$a(t)(T,u) := \eta'\left(\delta u - i\varphi(u)\frac{1-e^{-\lambda(T-t)}}{\lambda}\right)i\varphi(u)e^{-\lambda(T-t)}1_{[t,\infty)}(T),$$

cf. Section B.3.

By local independence of W,L,M (cf. Corollary A.13) we have

$$\begin{aligned}\psi_t^X(u) &= \psi_t^L(u) + \varphi(u)Z_{t-} + \eta(\delta u) - iu\eta(-\delta i)\\ &= \psi_0(t,u) + \int_0^t a(s)(t,u)ds + \varphi(u)Z_{t-}\\ &= \Psi_{t-}(t,u).\end{aligned}$$

The condition on the initial codebook implies that the mapping

$$(T,u)\mapsto \psi_0(T,u) + \int_0^T a(s)(T,u)ds - \gamma(u,0) = \psi^L(T,u)$$

is in $L^1(\mathbb{R}_+,\Pi)$. By Lemma 4.14 we have that φ_t from (4.20) is in $L^1(\mathbb{R}_+,\Pi)$ as well for any $t\in\mathbb{R}_+$. As in the proof of Corollary 4.12(2) it follows that Ψ_t has values in $L^1(\mathbb{R}_+,\Pi)$ for any $t\in\mathbb{R}_+$.

Thus $(X,\Psi_0,\alpha,\beta,M)$ with $\alpha_t(T,u) := a(t)(T,u)$, $\beta_t(T,u) := b(t)(T,u)$ is a compatible option surface model which satisfies the consistency condition and the drift condition (3.10). By Theorem 3.7 it is risk neutral, which yields the claim. □

Remark 4.17. Up to the additional time-inhomogeneous Lévy process L, the stock price model in (4.25, 4.26) is a special case of the so-called BNS model of [2]. If we consider more general functions φ, then, again up to the additional PII L, we end up with the CGMY extension of the BNS model from [11], cf. also [28].

5. CARMONA & NADTOCHIY'S 'TANGENT LÉVY MARKET MODELS'

In [10] and its extension [9, Section 5], Carmona and Nadtochiy (CN) developed independently a HJM-type approach for option prices with overlap to ours. Their simple model class in the sense of Step (4) in Section 2.1 is based on time-inhomogeneous Lévy processes as well. These can be described uniquely by their Lévy density and their diffusion coefficient because the drift is determined by the martingale condition for the stock under the risk neutral measure. Instead of the characteristic exponent from (2.4) CN use this Lévy density together with the diffusion coefficient as the codebook $(\kappa_t(T,x),\Sigma_t(T))$. Since we basically allow for the same class of simple models, their framework can be embedded into ours. Indeed, there is a transformation A that converts their codebook into ours, given by

$$A(\kappa_t(T,x),\Sigma_t(T)) := -\frac{u^2+iu}{2}\Sigma_t^2(T) + \int (e^{iux}-1-iu(e^x-1))\kappa_t(T,x)dx.$$

Since the simple models are parametrised differently, the drift condition in the two approaches differ. The condition in the CN framework looks a little more complex because it involves convolutions and differential operators of second order.

CN focus on Itô processes for modelling the codebook process, which roughly corresponds to choosing M as Brownian motion in our setup. Surprisingly, their approach leads to a constant diffusion coefficient $\Sigma_t(T)=\Sigma_0(T)$, cf. [9, Section 5]. This constant diffusion coefficient implies that the continuous martingale part of the stock price process follows a time-inhomogeneous Brownian motion rather than a more general continuous semimartingale. This phenomenon does not occur in our setup if the codebook is driven by a subordinator M, cf. e.g. Corollary 4.16.

With regards existence and uniqueness of models given basic building blocks, CN and we provide different answers. Our Theorems 4.7 and 4.9 imply existence and uniqueness for a subordinator M and given β_t is a sufficiently regular function of time and the current state of the codebook. By contrast, CN consider a different situation in their [10, Theorem 16] where they assume that the process β in their codebook dynamics

$$d\kappa_t = \alpha_t dt + \beta_t dB_t,$$

is given beforehand. This does not allow for the case that β depends on the current state κ of the codebook itself, which occurs e.g. in the example in Section 6 of [9] and is treated separately.

Both CN and we provide basically one non-trivial example, based on more or less deterministic β. In order to ensure existence of a compatible option surface model we assume the initial codebook to be large enough whereas CN slow down the codebook process when necessary.

Appendix A. Local characteristics and local exponents

In this section we define and recall some properties of local characteristics and local exponents.

A.1. Local characteristics.

Let X be an $\mathbb{R}^d$-valued semimartingale with integral characteristics (B,C,ν) in the sense of [25] relative to some fixed truncation function $h:\mathbb{R}^d\to\mathbb{R}^d$. By [25, I.2.9] there exist a predictable $\mathbb{R}^d$-valued process b, a predictable $\mathbb{R}^{d\times d}$-valued process c, a kernel K from $(\Omega\times\mathbb{R},\mathscr{P})$ to $(\mathbb{R}^d,\mathscr{B})$, and a predictable increasing process A such that

$$dB_t = b_t dA_t,\quad dC_t = c_t dA_t,\quad \nu(dt,dx) = K_t(dx)dA_t.$$

If $A_t = t$, we call the triplet (b,c,K) *local* or *differential characteristics* of X relative to truncation function h. Most processes in applications as e.g. diffusions, Lévy processes etc. allow for local characteristics. In this case b stands for a drift rate, c for a diffusion coefficient, and K for a local Lévy measure representing jump activity. If they exist, the local characteristics are unique up to a $dP\otimes dt$-null set on $\Omega\times\mathbb{R}_+$.

Proposition A.1 (Itô's formula for local characteristics). *Let X be an $\mathbb{R}^d$-valued semimartingale with local characteristics (b,c,K) and $f:\mathbb{R}^d\to\mathbb{R}^n$ a C^2-function. Then the triplet*

$(\widetilde{b},\widetilde{c},\widetilde{K})$ *defined by*

$$\begin{aligned}
\widetilde{b}_t &= Df(X_{t-})^\top b_t + \frac{1}{2}\sum_{j,k=1}^n \partial_j\partial_k f(X_{t-})c_t^{jk} \\
&\quad + \int \left(\widetilde{h}(f(X_{t-}+x)-f(X_{t-})) - (Df(X_{t-}))^\top h(x)\right) K_t(dx), \\
\widetilde{c}_t &= (Df(X_{t-}))^\top c_t Df(X_{t-}), \\
\widetilde{K}_t(A) &= \int 1_A(f(X_{t-}+x)-f(X_{t-}))K_t(dx), \quad A \in \mathscr{B}^n \text{ with } 0 \notin A,
\end{aligned}$$

is a version of the local characteristics of $f(X)$ with respect to a truncation function $\widetilde{h}$ on $\mathbb{R}^n$. Here, ∂_j etc. denote partial derivatives relative to the j'th argument.

Proof. See [28, Proposition 2.5].

□

Proposition A.2. *Let X be an $\mathbb{R}^d$-valued semimartingale with local characteristics (b,c,K) and let $\beta = (\beta^{ij})_{i\in\{1,\dots,d\},j\in\{1,\dots,n\}}$ be a $\mathbb{R}^{d\times n}$-valued predictable process such that $\beta^{\cdot i} \in L(X)$ for $i \in \{1,\dots,n\}$. Then the triplet $(\widetilde{b},\widetilde{c},\widetilde{K})$ defined by*

$$\begin{aligned}
\widetilde{b}_t &= \beta_t^\top b_t + \int \left(\widetilde{h}(\beta_t^\top x) - \beta_t^\top h(x)\right) K_t(dx), \\
\widetilde{c}_t &= \beta_t^\top c_t \beta_t, \\
\widetilde{K}_t(A) &= \int 1_A(\beta_t^\top x)K_t(dx), \quad A \in \mathscr{B}^n \text{ with } 0 \notin A,
\end{aligned}$$

is a version of the local characteristics of the $\mathbb{R}^n$-valued semimartingale $\beta \bullet X := (\beta^{\cdot 1} \bullet X,\dots,\beta^{\cdot n} \bullet X)$ with respect to the truncation function $\widetilde{h}$ on $\mathbb{R}^n$,

Proof. See [28, Proposition 2.4].

□

A.2. **Local exponents.**

Definition A.3. Let (b,c,K) be a Lévy-Khintchine triplet on $\mathbb{R}^d$ relative to some truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$. We call the mapping $\psi : \mathbb{R}^d \to \mathbb{C}$,

$$\psi(u) := iub - \frac{1}{2}u^\top cu + \int (e^{iux} - 1 - iuh(x))K(dx) \tag{A.1}$$

Lévy exponent corresponding to (b,c,K). By [25, II.2.44], the Lévy exponent determines the triplet (b,c,K) uniquely. If X is a Lévy process with Lévy-Khintchine triplet (b,c,K), we call ψ the *characteristic* or *Lévy exponent* of X. If (A.1) exists for all $u \in U \supset \mathbb{R}^d$, we call $\psi : U \to \mathbb{C}$ defined by (A.1) *extended Lévy exponent* of X on U.

In the same vein, local characteristics naturally lead to local exponents.

Definition A.4. If X is an $\mathbb{R}^d$-valued semimartingale with local characteristics (b,c,K), we write

$$\psi_t^X(u) := iub_t - \frac{1}{2}u^\top c_t u + \int (e^{iux} - 1 - iuh(x))K_t(dx), \quad u \in \mathbb{R}^d \tag{A.2}$$

for the Lévy exponent corresponding to (b_t,c_t,K_t). We call the family of predictable processes $\psi^X(u) := (\psi_t^X(u))_{t\in\mathbb{R}_+}, u \in \mathbb{R}^d$ *local exponent* of X. (A.2) implies that $u \mapsto \psi_t^X(u)$ is the characteristic exponent of a Lévy process.

The name *exponent* is of course motivated by the following fact.

Remark A.5. If X is a semimartingale with deterministic local characteristics (b,c,K), it is a PII and we have

$$E(e^{iu(X_T-X_t)}|\mathscr{F}_t)=E(e^{iu(X_T-X_t)})=\exp\left(\int_t^T \psi_s^X(u)ds\right)$$

for any $T\in\mathbb{R}_+$, $t\in[0,T]$, $u\in\mathbb{R}^d$, cf. [25, II.4.15].

We now generalize the notion of local exponents to complex-valued semimartingales and more general arguments.

Definition A.6. Let X be a $\mathbb{C}^d$-valued semimartingale and β a $\mathbb{C}^d$-valued X-integrable process. We call a predictable $\mathbb{C}$-valued process $\psi^X(\beta)=(\psi_t^X(\beta))_{t\in\mathbb{R}_+}$ *local exponent* of X at β if $\psi^X(\beta)\in L(I)$ and $(\exp(i\beta\bullet X_t-\int_0^t\psi_s^X(\beta)ds))_{t\in\mathbb{R}_+}$ is a complex-valued local martingale. We denote by $\mathscr{U}^X$ the set of processes β such that the local exponent $\psi^X(\beta)$ exists.

From the following lemma it follows that $\psi^X(\beta)$ is unique up to a $dP\otimes dt$-null set.

Lemma A.7. *Let X be a complex-valued semimartingale and A,B complex-valued predictable processes of finite variation with $A_0=0=B_0$ and such that* $\exp(X-A)$ *and* $\exp(X-B)$ *are local martingales. Then $A=B$ up to indistinguishability.*

Proof. Set $M:=e^{X-A}$, $N:=e^{X-B}$, $V:=e^{A-B}$. Integration by parts yields that

$$M_-\bullet V=MV-V\bullet M-M_0V_0=N-V\bullet M-M_0$$

is a local martingale. Therefore $V=1+\frac{1}{M_-}\bullet(M_-\bullet V)$ is a predictable local martingale with $V_0=1$ and hence $V=1$, cf. [25, I.3.16].

□

The following result shows that Definition A.6 truly generalizes Definition A.4.

Proposition A.8. *Let X be an $\mathbb{R}^d$-valued semimartingale with local characteristics (b,c,K). Suppose that β is a $\mathbb{C}^d$-valued predictable and X-integrable process. If β is $\mathbb{R}^d$-valued for any $t\in\mathbb{R}_+$, then $\beta\in\mathscr{U}^X$. Moreover there is equivalence between*

(1) $\beta\in\mathscr{U}^X$,

(2) $\int_0^t\int 1_{\{-\mathrm{Im}(\beta_s x)>1\}}e^{-\mathrm{Im}(\beta_s x)}K_s(dx)ds<\infty$ *almost surely for any $t\in\mathbb{R}_+$.*

In this case we have

$$\psi_t^X(\beta)=i\beta_t b_t-\frac{1}{2}\beta_t^\top c_t\beta_t+\int(e^{i\beta_t x}-1-i\beta_t h(x))K_t(dx) \tag{A.3}$$

outside some $dP\otimes dt$-null set.

Proof. If β is $\mathbb{R}^d$-valued, then Statement (2) is obviously true. Thus we only need to prove the equivalence and (A.3). For real-valued $i\beta$ the equivalence follows from [29, Lemma 2.13]. The complex-valued case is derived similarly. For real-valued $i\beta$ (A.3) is shown in [29, Theorems 2.18(1,6) and 2.19]. The general case follows along the same lines.

□

(A.3) implies that the local exponent of X at any $\beta\in\mathscr{U}^X$ is determined by the triplet (b,c,K) and hence by the local exponent of X in the sense of Definition A.4.

Corollary A.9. *Let (X,M) be a $1+d$-dimensional semimartingale with local exponent $\psi^{(X,M)}$ such that M is a Lévy process whose components are subordinators. Then $\beta\in\mathscr{U}^{(X,M)}$ for any $\mathbb{R}\times(\mathbb{R}+i\mathbb{R}_+)^d$-valued (X,M)-integrable process β.*

Proof. This follows immediately from Proposition A.8.

□

Definition A.10. Let $X^{(1)},\dots,X^{(n)}$ be semimartingales which allow for local characteristics. We call them $X^{(1)},\dots,X^{(n)}$ *locally independent* if

$$\mathscr{U}^{(X^{(1)},\dots,X^{(n)})} \cap (L(X^{(1)})\times\cdots\times L(X^{(n)})) = \mathscr{U}^{X^{(1)}}\times\cdots\times\mathscr{U}^{X^{(n)}}$$

for

$$\begin{aligned} L(X^{(1)})\times\cdots\times L(X^{(n)}) \quad := \quad & \{\beta=(\beta^{(1)},\dots,\beta^{(n)}) \text{ complex-valued}: \\ & \beta^{(i)}\ X^{(i)}\text{-integrable for } i=1,\dots,n\} \end{aligned}$$

and

$$\psi^{(X^{(1)},\dots,X^{(n)})}(\beta) = \sum_{j=1}^{n} \psi^{X^{(j)}}(\beta^{(j)})$$

outside some $dP\otimes dt$-null set for any $\beta=(\beta^{(1)},\dots,\beta^{(n)})\in\mathscr{U}^{(X^{(1)},\dots,X^{(n)})}$.

The following lemma provides alternative characterisations of local independence. For ease of notation we consider two semimartingales but the extension to arbitrary finite numbers is straightforward.

Lemma A.11. *Let (X,Y) be an $\mathbb{R}^{m+n}$-valued semimartingale with local characteristics (b,c,K) and denote by (b^X,c^X,K^X) resp. (b^Y,c^Y,K^Y) local characteristics of X resp. Y. We have equivalence between*

(1) *X and Y are locally independent,*

(2)
$$\psi^{(X,Y)}(u,v) = \psi^X(u)+\psi^Y(v), \quad (u,v)\in\mathbb{R}^{m+n} \tag{A.4}$$
outside some $dP\otimes dt$-null set,

(3)
$$c = \begin{pmatrix} c^X & 0 \\ 0 & c^Y \end{pmatrix}$$
and
$$K(A) = K^X(\{x:(x,0)\in A\})+K^Y(\{y:(0,y)\in A\}), \quad A\in\mathscr{B}^{m+n}$$
outside some $dP\otimes dt$-null set.

Proof. (1)⇒(2): This is obvious by Proposition A.8.

(2)⇒(3): Both sides of (A.4) are Lévy exponents for fixed $(\omega,t)\in\Omega\times\mathbb{R}_+$. Indeed, the triplet corresponding to $(u,v)\mapsto(\psi_t^X(u)+\psi_t^Y(v))(\omega)$ is $(b_t,\widetilde{c}_t,\widetilde{K}_t)(\omega)$ with

$$\widetilde{c}_t = \begin{pmatrix} c_t^X & 0 \\ 0 & c_t^Y \end{pmatrix}$$

and

$$\widetilde{K}_t(A) = K_t^X(\{x:(x,0)\in A\})+K_t^Y(\{y:(0,y)\in A\}), \quad A\in\mathscr{B}^{m+n}.$$

Since the Lévy exponent determines the triplet uniquely (cf. [25, II.2.44]), the assertion follows.

(3)⇒(1): If β^X is X-integrable and β^Y is Y-integrable, then $\beta=(\beta^X,\beta^Y)$ is (X,Y)-integrable and $\beta\bullet(X,Y)=\beta^X\bullet X+\beta^Y\bullet Y$. The characterisation in Proposition A.8 yields $\beta\in\mathscr{U}^{(X,Y)}$ for such $\beta=(\beta^X,\beta^Y)$ if and only if $\beta^X\in\mathscr{U}^X$, $\beta^Y\in\mathscr{U}^Y$. In addition, $\psi^{(X,Y)}(\beta)=\psi^X(\beta^X)+\psi^Y(\beta^Y)$ follows from (A.3)

□

Corollary A.12. *Let $X^{(1)},\dots,X^{(n)}$ be locally independent semimartingales and $Q \overset{\text{loc}}{\ll} P$ another probability measure. Then $X^{(1)},\dots,X^{(n)}$ are locally independent semimartingales relative to Q.*

Proof. This follows from Lemma A.11 and [25, III.3.24].

□

Corollary A.13. *If $(X^{(1)},\dots,X^{(n)})$ is a Lévy process or, more generally, a PII allowing for local characteristics, then $X^{(1)},\dots,X^{(n)}$ are independent if and only if they are locally independent.*

Proof. By Remark A.5 the characteristic function φ_{X_t} of $X_t := (X_t^{(1)},\dots,X_t^{(n)})$ is given by

$$\varphi_{X_t}(u^1,\dots,u^n) = \exp\left(\int_0^t \psi_s^{(X^{(1)},\dots,X^{(n)})}(u^1,\dots,u^n)ds\right).$$

Thus independence of $X_t^{(1)},\dots,X_t^{(n)}$ is equivalent to

$$\psi_t^{(X^{(1)},\dots,X^{(n)})}(u^1,\dots,u^n) = \sum_{k=1}^n \psi_t^{X^k}(u^k),$$

for Lebesgue-almost any $t \in \mathbb{R}_+$ and any $(u^1,\dots,u^n) \in \mathbb{R}^n$. By Lemma A.11 this in turn is equivalent to local independence of $X_t^{(1)},\dots,X_t^{(n)}$.

□

Lemma A.14. *If $\varphi \in \Pi$, then $\mathrm{Re}(\varphi(u)) \leq 0$ for any $u \in \mathbb{R}$, where Π is defined in Section 3.1.*

Proof. For $\varphi \in \Pi$ we have

$$\varphi(u) = -\frac{u^2+iu}{2}c + \int (e^{iux} - 1 - iu(e^x - 1))K(dx) \tag{A.5}$$

with some Lévy measure K and some $c \in \mathbb{R}_+$. The real part of the first term is obviously negative and the real part of the integrand is negative as well.

□

Remark A.15. If we extend the domain of φ to $\mathbb{R} + i[-1,0]$ by keeping the representation (A.5), then the conclusion of Lemma A.14 is still correct. However, this fact is not used in this paper.

The following four lemmas follow immediately from the definition of local exponents.

Lemma A.16. *Let X be a $\mathbb{C}$-valued semimartingale that allows for local characteristics. Then there is equivalence between*

(1) $\exp(X)$ *is a local martingale,*
(2) $-i \in \mathscr{U}^X$ *and* $\psi^X(-i) = 0$ *outside some* $dP \otimes dt$*-null set.*

Remark A.17. If X in Lemma A.16 is real-valued and if e^X is a local martingale, Proposition A.8 yields that the mapping $\mathbb{R} \to \mathbb{C}$, $u \mapsto \psi_t^X(u)$ is in Π outside some $dP \otimes dt$-null set.

Lemma A.18. *Let X be a $\mathbb{C}^d$-valued semimartingale, β a $\mathbb{C}^d$-valued and X-integrable process, and $u \in \mathbb{C}$. Then $u\beta \in \mathscr{U}^X$ if and only if $u \in \mathscr{U}^{\beta \bullet X}$. In that case we have*

$$\psi^X(u\beta) = \psi^{\beta \bullet X}(u).$$

Lemma A.19. *Let X,Y be $\mathbb{C}^d$-valued semimartingales and $u \in \mathbb{C}^d$. Then $u \in \mathscr{U}^{X+Y}$ if and only if $(u,u) \in \mathscr{U}^{(X,Y)}$. In this case we have*

$$\psi^{X+Y}(u) = \psi^{(X,Y)}(u,u)$$

outside some $dP \otimes dt$-null set.

Lemma A.20. *Let X,Z be $\mathbb{C}^d$-valued semimartingales and β,γ predictable $\mathbb{C}^d$-valued processes such that*

(1) *γ has I-integrable components,*
(2) *$\beta\gamma$ is I-integrable,*
(3) *$Z_t = Z_0 + \int_0^t \gamma_s ds + X_t$.*

Then $\beta \in \mathscr{U}^Z$ if and only if $\beta \in \mathscr{U}^X$. In this case $\psi^Z(\beta) = \psi^X(\beta) + i\beta\gamma$ outside some $dP \otimes dt$-null set.

Proposition A.21. *Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$ be a filtered proability space and $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$ a sub-filtration of $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$. Moreover, let X be an $\mathbb{R}^d$-valued semimartingale with integrable characteristics (B,C,ν) such that X is $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$-adapted and $B,C,\nu(A)$ are $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$-predictable for any $A \in \mathscr{B}(\mathbb{R}^d)$. Then X is a semimartingale with integral characteristics (B,C,ν) relative to filtration $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$as well.*

Proof. Let $u \in \mathbb{R}^d$ and define the process

$$A(u)_t := iuB_t - \frac{1}{2}u^\top C_t u + \int (e^{iux} - 1 - iuh(x))\nu_t(dx).$$

[25, Theorem II.2.42] yields that $e^{iuX} - e^{iuX_-} \bullet A(u)$ is an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-local martingale. The process $Y := e^{iuX_-} \bullet A(u)$ is a $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$-predictable process by assumption with càdlàg paths. This implies that it is $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$-locally bounded because it is càdlàg and predictable, cf. [26, Lemma A.1]. If $(\tau_n)_{n\in\mathbb{N}}$ denotes a corresponding sequence of stopping times, then $(e^{iuX} - Y)^{\tau_n}$ is bounded and hence it is a $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-martingale. [23, Corollaire 9.16] yields that it is a $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$-martingale. Therefore $e^{iuX} - Y$ is a $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$-local martingale. Consequently, [25, Theorem II.2.42] yields that X is a $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$-semimartingale with integral characteristics (B,C,ν).

□

A.3. **Semimartingale decomposition relative to a semimartingale.** Let (X,Y) be an $\mathbb{R}^{1+d}$-valued semimartingale with local characteristics (b,c,K), written here in the form

$$b = \begin{pmatrix} b^X \\ b^Y \end{pmatrix}, \quad c := \begin{pmatrix} c^X & c^{X,Y} \\ c^{Y,X} & c^Y \end{pmatrix}. \tag{A.6}$$

Suppose that $\int_0^t \int_{(1,\infty)} e^x K_s(dx)ds < \infty$ for any $t \in \mathbb{R}_+$ or, equivalently, e^X is a special semimartingale. We set

$$X_t^{\parallel} \;:=\; \log \mathscr{E}\Big((c^{X,Y}(c^Y)^{-1}) \bullet Y_t^c + f * (\mu^{(X,Y)} - \nu^{(X,Y)})_t\Big)$$

for any $t \in \mathbb{R}_+$, where c^- denotes the pseudoinverse of a matrix c in the sense of [1], Y^c is the continuous local martingale part of Y, $\mu^{(X,Y)}$ resp. $\nu^{(X,Y)}$ are the random measure of jumps of (X,Y) and its compensator, and $f : \mathbb{R}^{1+d} \to \mathbb{R}, (x,y) \mapsto (e^x - 1)1_{\{y\neq 0\}}$. We call $X^{\parallel}$ and $X^{\perp} := X - X^{\parallel}$ the *dependent* resp. *independent part* of X relative to Y.

Lemma A.22. *$X \mapsto X^{\parallel}$ is a projection in the sense that $(X^{\parallel})^{\parallel} = X^{\parallel}$. Moreover, we have $(X+Z)^{\parallel} = X^{\parallel}$ if Z is a semimartingale such that Z,Y are locally independent.*

Proof. Observe that $(X^{\|})^c = (c^{X,Y}(c^Y)^{-1}) \bullet Y^c$ by [29, Lemma 2.6(2)]. Defining $c^{X^{\|},Y}$ similarly as $c^{X,Y}$ in (A.6), we have $c^{X^{\|},Y} = c^{X,Y}(c^Y)^{-1}c^Y = c^{X,Y}$. Moreover, $f(\Delta X^{\|}_t, \Delta Y_t) = f(\Delta X_t, \Delta Y_t)$ for any $t \geq 0$, which implies $f * (\mu^{(X^{\|},Y)} - \nu^{(X^{\|},Y)}) = f * (\mu^{(X,Y)} - \nu^{(X,Y)})$ by definition of stochastic integration relative to compensated random measures. Together, the first assertion follows.

Using the notation of (A.6), note that $c^{(X+Z),Y} = c^{X,Y} + c^{Z,Y} = c^{X,Y}$ by Lemma A.11. Lemma A.11 also implies that Z and Y do not jump together (outside some evanescent set) and hence $f(\Delta(X+Z)_t, \Delta Y_t) = f(\Delta X_t, \Delta Y_t)$. This implies $f * (\mu^{(X^{\|},Y)} - \nu^{(X^{\|},Y)}) = f * (\mu^{(X,Y)} - \nu^{(X,Y)})$ and hence $(X+Z)^{\|} = X^{\|}$.

□

Lemma A.23. *$e^{X^{\|}}$ is a local martingale. Moreover, $X^{\perp}$ and $(X^{\|},Y)$ are locally independent semimartingales. Finally, $e^{X^{\perp}}$ is a local martingale if and only if e^X is a local martingale.*

Proof. The first statement is obvious. The last statement follows from the first two and from Lemma A.16. It remains to prove local independence of $X^{\perp}$ and $(X^{\|},Y)$. Denote the local characteristics of $(X^{\perp},X^{\|},Y)$ by $(b^{(X^{\perp},X^{\|},Y)}, c^{(X^{\perp},X^{\|},Y)}, K^{(X^{\perp},X^{\|},Y)})$ and accordingly for $(X,X^{\|},Y)$, $X^{\perp}$ etc. Set $\bar{c} := c^{X,Y}(c^Y)^{-1}c^{Y,X}$. Since $(X^{\|})^c = (c^{X,Y}(c^Y)^{-1}) \bullet Y^c$, we have

$$c^{(X,X^{\|},Y)} = \begin{pmatrix} c^X & \bar{c} & c^{X,Y} \\ \bar{c} & \bar{c} & c^{X,Y} \\ c^{Y,X} & c^{Y,X} & c^Y \end{pmatrix}$$

and hence

$$c^{(X^{\perp},X^{\|},Y)} = \begin{pmatrix} c^X - \bar{c} & 0 & 0 \\ 0 & \bar{c} & c^{X,Y} \\ 0 & c^{Y,X} & c^Y \end{pmatrix}$$

e.g. by Proposition A.1. Moreover,

$$\begin{aligned} \Delta(X^{\perp},X^{\|},Y)_t &= 1_{\{\Delta Y_t = 0\}}(\Delta X_t, 0, 0) + 1_{\{\Delta Y_t \neq 0\}}(0, \Delta X_t, \Delta Y_t) \\ &= \begin{cases} (\Delta X^{\perp}_t, 0, 0) & \text{if } \Delta X^{\perp}_t \neq 0, \\ (0, \Delta(X^{\|},Y)_t) & \text{if } \Delta(X^{\|},Y)_t \neq 0, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

yields

$$K^{(X^{\perp},X^{\|},Y)}(A) = K^{X^{\perp}}(\{x : (x,0,0) \in A\}) + K^{(X^{\|},Y)}(\{(x,z) : (0,x,z) \in A\})$$

for $A \in \mathscr{B}^{2+d}$. Lemma A.11 completes the proof.

□

Appendix B. Technical Proofs

B.1. **Option pricing by Fourier transform.** By

$$\mathscr{F}f(u) := \lim_{C \to \infty} \int_{-C}^{\infty} f(x) e^{iux} dx \tag{B.1}$$

we denote the (left-)improper Fourier transform of a measurable function $f : \mathbb{R} \to \mathbb{C}$ for any $u \in \mathbb{R}$ such that the expression exists. If f is Lebesgue integrable, then the improper Fourier transform and the ordinary Fourier transform (i.e. $u \mapsto \int f(x) e^{iux} dx$) coincide. In our application in Section 3 the improper Fourier transform exists for any $u \in \mathbb{R} \setminus \{0\}$. Moreover, we denote by

$$\mathscr{F}^{-1}g(x) := \frac{1}{2\pi}\left(\lim_{\varepsilon\downarrow 0}\int_{\varepsilon}^{\infty} e^{-iux}g(u)du + \lim_{\varepsilon\downarrow 0}\int_{-\infty}^{-\varepsilon} e^{-iux}g(u)du\right) \tag{B.2}$$

an improper inverse Fourier transform, which is suitable to our application in Section 3.

Lemma B.1. *Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\mathscr{G} \subset \mathscr{F}$ a sub-σ-field. Furthermore suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is $\mathscr{F} \otimes \mathscr{B}$-measurable, $m : \Omega \to \mathbb{R}$ is $\mathscr{F}$-measurable and*

$$H(x) := 1_{[0,m]}(x)f(x) - 1_{[m,0)}(x)f(x)$$

is nonnegative with $E(\int_0^m f(x)dx) < \infty$. (Note that $\int_0^m f(x)dx$ is always nonnegative.) Then we have

$$\mathscr{F}\{x \mapsto E(H(x)|\mathscr{G})\}(u) = E\left(\int_0^m f(x)e^{iux}dx\middle|\mathscr{G}\right), \quad u \in \mathbb{R} \tag{B.3}$$

where the improper Fourier transform coincides with the ordinary Fourier transform.

Proof. Let $u \in \mathbb{R}$. From

$$\begin{aligned}\int_0^{\infty} H(x)e^{iux}dx &= 1_{\{m\geq 0\}}\int_0^m f(x)e^{iux}dx,\\ \int_{-\infty}^{0} H(x)e^{iux}dx &= 1_{\{m<0\}}\int_0^m f(x)e^{iux}dx\end{aligned}$$

it follows that $\int_{-\infty}^{\infty} H(x)e^{iux}dx = \int_0^m f(x)e^{iux}dx$. This implies

$$E\left(E\left(\int_{-\infty}^{\infty} H(x)dx\middle|\mathscr{G}\right)\right) = E\left(\int_{-\infty}^{\infty} H(x)dx\right) = E\left(\int_0^m f(x)dx\right) < \infty$$

and hence

$$\int_{-\infty}^{\infty} E(H(x)|\mathscr{G})dx = E\left(\int_{-\infty}^{\infty} H(x)dx\middle|\mathscr{G}\right) < \infty.$$

Now we can apply Fubini's theorem and we get

$$\mathscr{F}\{x \mapsto E(H(x)|\mathscr{G})\}(u) = E\left(\int_{-\infty}^{\infty} H(x)e^{iux}dx\middle|\mathscr{G}\right) = E\left(\int_0^m f(x)e^{iux}dx\middle|\mathscr{G}\right).$$

□

The next proposition is a modification of [4, Proposition 1], cf. also [12].

Lemma B.2. *Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\mathscr{G} \subset \mathscr{F}$ a sub-σ-field. Let Y be a random variable such that $E(e^Y) < \infty$ and consider*

$$\mathscr{O}(x) := \begin{cases} E((e^{Y-x}-1)^+|\mathscr{G}) & \text{if } x \geq 0,\\ E((1-e^{Y-x})^+|\mathscr{G}) & \text{if } x < 0.\end{cases}$$

Then we have

$$\mathscr{F}\{x \mapsto \mathscr{O}(x)\}(u) = \frac{1}{iu} - \frac{E(e^Y|\mathscr{G})}{iu-1} - \frac{E(e^{iuY}|\mathscr{G})}{u^2+iu}$$

and

$$\begin{aligned}&\mathscr{F}\{x \mapsto 1_{\{x\geq -C\}}\mathscr{O}(x)\}(u)\\ &= \frac{1}{iu} - \frac{E\left(e^y|\mathscr{G}\right)}{iu-1} - \frac{1 - E\left(e^{iu(Y\vee -C)}\left(1+iu\left(e^{0\wedge(Y+C)}-1\right)\right)\middle|\mathscr{G}\right)}{u^2+iu}\end{aligned}$$

for any $C \in \mathbb{R}_+$, $u \in \mathbb{R} \setminus \{0\}$. *If* $E(e^Y|\mathscr{G}) = 1$, *then in particular*

$$\mathscr{F}\{x \mapsto \mathscr{O}(x)\}(u) = \frac{1 - E(e^{iuY}|\mathscr{G})}{u^2 + iu}$$

for any $u \in \mathbb{R} \setminus \{0\}$.

Proof. Let $C \in \mathbb{R}_+$, $u \in \mathbb{R} \setminus \{0\}$. We define $m := (Y \vee -C)$, $f(x) := e^{Y-x} - 1$, and $H(x) := 1_{[0,m]}(x) f(x) - 1_{[m,0)}(x) f(x)$. Then we have $1_{\{x \geq -C\}} \mathscr{O}(x) = E(H(x)|\mathscr{G})$, $H \geq 0$, and

$$E\left(\int_0^m f(x)dx\right) = E(e^Y - m - e^{Y-m}) < \infty.$$

Hence Lemma B.1 yields

$$\begin{aligned}
\int_{-C}^{\infty} \mathscr{O}(x) e^{iux} dx &= \mathscr{F}\{x \mapsto E(H(x)|\mathscr{G})\}(u) \\
&= E\left(\int_0^m (e^{Y-x} - 1) e^{iux} dx \middle| \mathscr{G}\right) \\
&= E\left(\left[\frac{e^{Y+(iu-1)x}}{iu-1} - \frac{e^{iux}}{iu}\right]_{x=0}^{m} \middle| \mathscr{G}\right) \\
&= \frac{1}{iu} - \frac{E(e^Y|\mathscr{G})}{iu-1} - \frac{E\left(e^{ium}(1 + iu(e^{Y-m} - 1))|\mathscr{G}\right)}{u^2 + iu}.
\end{aligned}$$

Since $|e^{ium}(1 + iu(e^{Y-m} - 1))| \leq 1 + |u|$, we can apply Lebesgue's theorem and get

$$E\left(e^{ium}(1 + iu(e^{Y-m} - 1))|\mathscr{G}\right) \overset{C \to \infty}{\longrightarrow} E\left(e^{iuY}|\mathscr{G}\right).$$

□

Corollary B.3. *Let* Y *be a random variable with* $E(e^Y) < \infty$. *Define*

$$O(x) := \begin{cases} E\left((e^{Y-x} - 1)^+\right) & \text{if } x \geq 0, \\ E\left((1 - e^{Y-x})^+\right) & \text{if } x < 0. \end{cases}$$

Then we have

$$\left|\int_{-C}^{\infty} O(x) e^{iux} dx\right| \leq E(e^Y) + \frac{1 + 2|u|}{u^2}$$

for any $C \in \mathbb{R}_+$, $u \in \mathbb{R} \setminus \{0\}$.

Proof. This follows from the second statement of Lemma B.2.

□

Proposition B.4. *Let* $(\Omega, \mathscr{F}, P)$ *be a probability space and* $\mathscr{G} \subset \mathscr{F}$ *a sub-*σ*-field. Let* Y *be a random variable with* $E(e^Y|\mathscr{G}) = 1$ *and define*

$$\mathscr{O}(x) := \begin{cases} E((e^{Y-x} - 1)^+|\mathscr{G}) & \text{if } x \geq 0, \\ E((1 - e^{Y-x})^+|\mathscr{G}) & \text{if } x < 0. \end{cases}$$

Then we have

$$\begin{aligned}
\mathscr{O}(x) &= \mathscr{F}^{-1}\left\{u \mapsto \frac{1 - E(e^{iuY}|\mathscr{G})}{u^2 + iu}\right\}(x), \\
E(e^{iuY}|\mathscr{G}) &= 1 - (u^2 + iu)\mathscr{F}\{x \mapsto \mathscr{O}(x)\}(u)
\end{aligned}$$

for any $u, x \in \mathbb{R}$.

Proof. The second equation is a restatement of Lemma B.2. Let $0<\alpha<1$ and define $O(x):=e^{\alpha x}\mathscr{O}(x)$, $m:=Y$, $f(x):=e^{\alpha x}(e^{Y-x}-1)$, and $H(x):=1_{[0,m]}(x)f(x)-1_{[m,0)}(x)f(x)$ for any $x\in\mathbb{R}$. Then H is nonnegative, $O(x)=E(H(x)|\mathscr{G})$, and

$$E\left(\int_0^m f(x)dx\right)=E\left(\frac{e^{\alpha Y}}{\alpha^2-\alpha}+\frac{e^Y}{1-\alpha}+\frac{1}{\alpha}\right)<\infty.$$

Lemma B.1 yields

$$\begin{aligned}\mathscr{F}\{x\mapsto O(x)\}(u) &= E\left(\int_0^m f(x)e^{iux}dx\middle|\mathscr{G}\right)\\ &= \frac{E\left(e^{(\alpha+iu)Y}\middle|\mathscr{G}\right)-1}{(\alpha+iu)^2-(\alpha+iu)}.\end{aligned}$$

We have $E(|e^{(\alpha+iu)Y}|)\leq E(1+e^Y)=2$ and thus $u\mapsto\mathscr{F}\{x\mapsto O(x)\}(u)$ is integrable. The Fourier inversion theorem yields

$$O(x)=\mathscr{F}^{-1}\{u\mapsto\mathscr{F}\{\tilde{x}\mapsto O(\tilde{x})\}(u)\}(x)$$

because the ordinary inverse Fourier transform coincides with the improper inverse Fourier transform for Lebesgue-integrable functions. Define

$$g:\{z\in\mathbb{C}\setminus\{0\}:-1<\mathrm{Re}(z)\leq 0\}\to\mathbb{C},\quad z\mapsto\frac{E(e^{-zY}|\mathscr{G})-1}{z^2+z}.$$

g is continuous and holomorphic in the interior of its domain. Let $0<\varepsilon<\frac{1}{2}=:\alpha$ and define

$$\begin{aligned}\gamma_{(1,\varepsilon)}:[-1,1]\to\mathbb{C},&\quad t\mapsto i\frac{t}{\varepsilon}-\frac{1}{2},\\ \gamma_{(2,\varepsilon)}:[0,1]\to\mathbb{C},&\quad t\mapsto i\frac{1}{\varepsilon}-\frac{1-t}{2},\\ \gamma_{(3,\varepsilon)}:[0,1]\to\mathbb{C},&\quad t\mapsto i(1-t)\left(\frac{1}{\varepsilon}-\varepsilon\right)+i\varepsilon,\\ \gamma_{(4,\varepsilon)}:[0,\pi]\to\mathbb{C},&\quad t\mapsto i\varepsilon e^{it},\\ \gamma_{(5,\varepsilon)}:[0,1]\to\mathbb{C},&\quad t\mapsto it\left(\varepsilon-\frac{1}{\varepsilon}\right)-i\varepsilon,\\ \gamma_{(6,\varepsilon)}:[0,1]\to\mathbb{C},&\quad t\mapsto -i\frac{1}{\varepsilon}-\frac{t}{2}\end{aligned}$$

as well as $\Gamma_\varepsilon:=\sum_{k=1}^6\gamma_{(k,\varepsilon)}$. Cauchy's integral theorem yields

$$\int_{\Gamma_\varepsilon}g(z)e^{xz}dz=0.$$

Moreover we have

$$\frac{1}{2\pi i}\int_{\gamma_{(1,\varepsilon)}}g(z)e^{xz}dz\overset{\varepsilon\to 0}{\longrightarrow}O(x)e^{-\frac{1}{2}x}=\mathscr{O}(x)$$

and

$$\int_{\gamma_{(k,\varepsilon)}}g(z)e^{xz}dz\overset{\varepsilon\to 0}{\longrightarrow}0$$

for $k\in\{2,6\}$ and even for $k=4$ because $zg(z)\to 0$ for $z\to 0$. Thus we conclude

$$\frac{1}{2\pi}\left(\int_{-1/\varepsilon}^{-\varepsilon}g(-iu)e^{-iux}du+\int_\varepsilon^{1/\varepsilon}g(-iu)e^{-iux}du\right)$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_{-\gamma_{(3,\varepsilon)}-\gamma_{(5,\varepsilon)}} g(z)e^{xz}dz \\
&= \frac{1}{2\pi i}\int_{\gamma_{(1,\varepsilon)}-\Gamma_\varepsilon+\gamma_(2,\varepsilon)+\gamma_(4,\varepsilon)+\gamma_(6,\varepsilon)} g(z)e^{xz}dz \\
&\overset{\varepsilon\to 0}{\longrightarrow} \mathscr{O}(x).
\end{aligned}
$$

Since

$$\int_{-\infty}^{-1/\varepsilon} g(-iu)e^{-iux}du+\int_{1/\varepsilon}^{\infty} g(-iu)e^{-iux}du \overset{\varepsilon\to 0}{\longrightarrow} 0,$$

we have

$$\frac{1}{2\pi}\left(\int_{-\infty}^{-\varepsilon} g(-iu)e^{-iux}du+\int_{\varepsilon}^{\infty} g(-iu)e^{-iux}du\right) \overset{\varepsilon\to 0}{\longrightarrow} \mathscr{O}(x)$$

and hence

$$\mathscr{F}^{-1}\left\{u\mapsto \frac{1-E(e^{iuY}|\mathscr{G})}{u^2+iu}\right\}(x)=\mathscr{O}(x).$$

□

Proposition B.5. *Let $(N(x))_{x\in\mathbb{R}}$ be a family of nonnegative local martingales and $(\tau_n)_{n\in\mathbb{N}}$ a common localising sequence for all $N(x)$ such that*

(1) $(\omega,x)\mapsto N_t(x)(\omega)$ *is $\mathscr{F}\otimes\mathscr{B}$-measurable for all $t\in\mathbb{R}_+$,*
(2) $x\mapsto N_t(x)(\omega)$ *is right-continuous and $\int_{-C}^{\infty} N_t(x)(\omega)dx<\infty$ for all $t\in\mathbb{R}_+$,*
(3) $\lim_{C\to\infty}\int_{-C}^{\infty} e^{iux}N_t(x)(\omega)dx$ *exists for all $\omega\in\Omega$, $t\in\mathbb{R}_+$, $u\in\mathbb{R}\setminus\{0\}$,*
(4) *for any $n\in\mathbb{N},t\in\mathbb{R},u\in\mathbb{R}\setminus\{0\}$ there is an integrable random variable Z such that*

$$\left|\int_{-C}^{\infty} e^{iux}N_t^{\tau_n}(x)(\omega)dx\right|\le Z$$

for any $C\in\mathbb{R}_+$.

Define the (improper, cf. (B.1)) Fourier transform of N by

$$X_t(u):=\mathscr{F}\{x\mapsto N_t(x)\}(u).$$

If $X(u)$ has càdlàg paths, then it is a local martingale for all $u\in\mathbb{R}\setminus\{0\}$ with common localising sequence $(\tau_n)_{n\in\mathbb{N}}$.

Proof. For any $C\in\mathbb{R}_+$, $u\in\mathbb{R}\setminus\{0\},\omega\in\Omega,t\in\mathbb{R}_+$ define

$$X_t^C(u)(\omega):=\int_{-C}^{\infty} e^{iux}N_t(x)(\omega)dx.$$

Fix $n\in\mathbb{N},C\in\mathbb{R}_+$, $u\in\mathbb{R}\setminus\{0\}$. Then $X^C_{t\wedge\tau_n(\omega)}(u)(\omega)=\int_{-C}^{\infty}e^{iux}N_{t\wedge\tau_n(\omega)}(x)(\omega)dx$ for any $t\in\mathbb{R}_+$, $\omega\in\Omega$. Setting

$$
\begin{aligned}
c(k,x) &:= \begin{cases} 1_{\cos(x)>0}\cos(x) & \text{for } k=0,\\ 1_{\sin(x)>0}\sin(x) & \text{for } k=1,\\ -1_{\cos(x)<0}\cos(x) & \text{for } k=2,\\ -1_{\sin(x)<0}\sin(x) & \text{for } k=3,\end{cases}\\
I_t^C(k) &:= \int_{-C}^{\infty} c(k,ux)N_{t\wedge\tau_n}(x)dx
\end{aligned}
$$

yields

$$X^C_{t\wedge\tau_n}(u)=\sum_{k=0}^{3} i^k I_t^C(k).$$

Since $c(k,\cdot):\mathbb{R}\to\mathbb{R}_+$ and hence $I^C(k)$ are positive, we can apply Tonelli's theorem and conclude that $I^C(k)$ is a martingale up to the càdlàg property for all $k\in\{0,1,2,3\}$. Thus $(t,\omega)\mapsto X^C_{t\wedge\tau_n}(u)(\omega)$ is a martingale up to the càdlàg property as well. By the definitions of X^C and X we have $X_t(u)(\omega)=\lim_{C\to\infty}X^C_t(u)(\omega)$ and thus we get

$$X_{t\wedge\tau_n(\omega)}(u)(\omega)=\lim_{C\to\infty}X^C_{t\wedge\tau_n(\omega)}(u)(\omega).$$

The fourth assumption on N and Lebesgue's theorem yield

$$\begin{aligned}
E(X_{t\wedge\tau_n}(u)|\mathscr{F}_s) &= E\left(\lim_{C\to\infty}X^C_{t\wedge\tau_n}(u)\middle|\mathscr{F}_s\right)\\
&= \lim_{C\to\infty}E\left(X^C_{t\wedge\tau_n}(u)\middle|\mathscr{F}_s\right)\\
&= X_{s\wedge\tau_n}(u)
\end{aligned}$$

for $s\leq t$. Thus $(t,\omega)\to X_t(u)(\omega)$ is a local martingale.

□

B.2. **Essential infimum of a subordinator.**

Definition B.6. Let ψ be a deterministic local exponent (of some semimartingale X). We define the *infimum process* E of ψ by $E_t:=\operatorname{ess\,inf}X_t$ for any $t\in\mathbb{R}_+$.

By [25, III.2.16], X in the previous definition is a PII whose law is determined by ψ. Since E_t is in turn determined by the law of X_t, the infimum process E does not depend on the particular choice of X.

Lemma B.7. *Let X,Y be independent random variables. Then*

$$\operatorname{ess\,inf}(X+Y)=\operatorname{ess\,inf}\ X+\operatorname{ess\,inf}\ Y.$$

Proof. Let $x:=\operatorname{ess\,inf}\ X$ and $y:=\operatorname{ess\,inf}\ Y$. We obviously have $\operatorname{ess\,inf}(X+Y)\geq x+y$ because $X+Y\geq x+y$ almost surely. Independence yields

$$\begin{aligned}
P(X+Y\leq x+y+\varepsilon) &\geq P\left(X\leq x+\frac{\varepsilon}{2},Y\leq y+\frac{\varepsilon}{2}\right)\\
&= P\left(X\leq x+\frac{\varepsilon}{2}\right)P\left(Y\leq y+\frac{\varepsilon}{2}\right)\\
&> 0
\end{aligned}$$

for any $\varepsilon>0$.

□

Proposition B.8. *Let X be a subordinator (i.e. an increasing Lévy process) and $E_t:=\operatorname{ess\,inf}X_t$ for any $t\in\mathbb{R}_+$. Then $E_t=tE_1\geq 0$ for any $t\geq 0$ and E is the drift part of X relative to the "truncation" function $h=0$. Moreover, $X-E$ is a subordinator.*

Proof. Since X is a subordinator, we have $E_t\geq 0$ for any $t\in\mathbb{R}_+$. Moreover, $\operatorname{ess\,inf}(X_t-X_s)=E_{t-s}$ because X_{t-s} has the same distribution as X_t-X_s. Since X_s and X_t-X_s are independent, Lemma B.7 yields $E_s+E_{t-s}=E_t$. The mapping $t\mapsto E_t$ is increasing because X is a subordinator. Together we conclude $E_t=tE_1$. This implies that $X-E$ is a positive Lévy process and hence a subordinator. By [34, Theorem 21.5] the Lévy-Khintchine triplet (b,c,K) relative to "truncation" function $h=0$ exists and satisfies $c=0$. Moreover, K and the random measure of jumps μ^X of X are concentrated on $\mathbb{R}_+$. In view of [25, II.2.34] we have $X_t=x*\mu^X_t+bt$ and thus we get $E_t=\operatorname{ess\,inf}X_t\geq bt$. According to [34, Theorem 21.5],

$X - E$ is a subordinator only if its drift rate $\widetilde{b}$ relative to $h = 0$ is greater or equal 0. Hence $bt - E_t = \widetilde{b}t \geq 0$, which implies $bt = E_t$.

□

Corollary B.9. *Let X be a d-dimensional semimartingale whose components X^k are subordinators with essential infimum $E^k_t = \text{ess inf}\, X^k_t$ for $k = 1,\dots,d$. For componentwise nonnegative bounded predictable processes φ we have*

$$\text{ess inf}\,(\varphi \bullet (X - E)_t) = 0.$$

Moreover, for bounded predictable $\mathbb{C}^d$-valued processes φ we have

$$\text{ess inf}\,|\varphi \bullet (X - E)_t| = 0$$

for any $t \in \mathbb{R}_+$.

Proof. The second statement is an application of the first statement because

$$0 \leq |\varphi \bullet (X - E)_t| \leq (|\varphi^1|,\dots,|\varphi^d|) \bullet (X - E)_t.$$

Suppose that φ is a componentwise nonnegative bounded predictable process. Proposition B.8 yields that $X^k - E^k$ is a subordinator with essential infimum 0. Hence we may assume w.l.o.g. that $E^k = 0$. Since φ is bounded, there is a constant $c \in \mathbb{R}_+$ such that $\varphi^k \leq c$ for $k = 1,\dots,d$. Hence $\varphi \bullet X_t \leq c \sum_{k=1}^d X^k_t$ for any $t \in \mathbb{R}_+$. By Proposition B.8 the drift part of X^k is 0 relative to the truncation function $h = 0$. Consequently, the drift part of the subordinator $L := c \sum_{k=1}^d X^k$ is also 0 relative to the truncation function $h = 0$. Proposition B.8 yields ess inf $L_t = 0$ for any $t \in \mathbb{R}_+$. Thus we conclude

$$0 \leq \text{ess inf}\,(\varphi \bullet X_t) \leq \text{ess inf}\, L_t = 0.$$

□

B.3. **Differentiability of Lévy exponents.**

Lemma B.10. *Let X denote an $\mathbb{R}^d$-valued Lévy process with finite second moments in the sense that $E(|X_t|^2) < \infty$ for some (and hence for any) $t > 0$. Then its Lévy exponent is twice continuously differentiable with bounded second-order derivatives.*

Proof. If (b, c, K) denotes the Lévy-Khintchine triplet of X, the Lévy exponent ψ of X is of the form (A.1). The existence of second moments yields $\int |x|^2 K(dx) < \infty$ by [34, Corollary 25.8, Definition 8.2]. Dominated convergence implies that we may differentiate under the integral sign and obtain

$$\partial_i \psi(u) = ib_i - c^{i\cdot} u + \int \big(ix_i e^{iux} - ih_i(x)\big) K(dx), \quad i = 1,\dots,d \tag{B.4}$$

and

$$\partial_i \partial_j \psi(u) = -c^{ij} - \int x_i x_j e^{iux} K(dx), \quad i, j = 1,\dots,d,$$

where ∂_i denotes the partial derivative relative to u_i and likewise for j. Since

$$2 \int |x_i x_j e^{iux}| K(dx) \leq \int x_i^2 K(dx) + \int x_j^2 K(dx),$$

the claim follows.

□

Remark B.11. Consider an $\mathbb{R}^2$-valued Lévy process X, M such that M is a subordinator. If X and M have finite second moments, the statement of the previous lemma holds also for its extended Lévy exponent $\psi^{(X,M)}$ on $\mathbb{R} \times (\mathbb{R} + i\mathbb{R}_+)$. This is shown along the same lines as above. Moreover, $\partial_2 \psi^{(X,M)}$ is bounded in this case by (B.4) and since $\int |x_2| K(dx) < \infty$.

B.4. **Existence of PII.**

Theorem B.12. *Let $\psi \in L^1(\mathbb{R}_+, E)$ such that $u \mapsto \int_t^T \psi(r,u)dr$ is a Lévy exponent for any $t,T \in \mathbb{R}$, $t \leq T$. Then there is a PII X such that $Ee^{iuX_t} = \exp(\int_0^t \psi(r,u)dr)$ for any $t \in \mathbb{R}_+$, $u \in \mathbb{R}$.*

Proof. Kolmogorov's extension theorem [34, Theorem 1.8] yields that there is a stochastic process Y with $Y_0 = 0$ a.s. and $Ee^{iu(Y_t - Y_s)} = \exp(\int_s^t \psi(r,u)dr)$ for any $s,t \in \mathbb{R}_+$ with $s \leq t$, $u \in \mathbb{R}$. This process satisfies (1) and (2) of [34, Definition 1.6]. We show that it is an additive process in law in the sense of [34, Definition 1.6], i.e. it is in addition stochasically continuous. Let $\varepsilon > 0$ and $s \in \mathbb{R}_+$. Let $\varphi : \mathbb{R} \to [0,1]$ be infinitely differentiable, its support contained in $[-\varepsilon, \varepsilon]$, and $\varphi(0) = 1$. Define

$$\check{\varphi} : \mathbb{R} \to \mathbb{C}, u \mapsto \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(x) e^{-iux} dx.$$

Then $\varphi(x) = \int_{-\infty}^{\infty} \check{\varphi}(u) e^{iux} du$ for any $x \in \mathbb{R}$. For $t > s$ we have

$$\begin{aligned} P(|Y_t - Y_s| > \varepsilon) &\leq 1 - E(\varphi(Y_t - Y_s)) \\ &= 1 - \int_{-\infty}^{\infty} \check{\varphi}(u) E e^{iu(Y_t - Y_s)} du \\ &= 1 - \int_{-\infty}^{\infty} \check{\varphi}(u) \exp\left(\int_s^t \psi(r,u)dr \right) du \\ &\underset{t \downarrow s}{\longrightarrow} 1 - \int_{-\infty}^{\infty} \check{\varphi}(u) du \\ &= 1 - \varphi(0) = 0, \end{aligned}$$

where $t > s$ and the convergence follows from the dominated convergence theorem. Similar arguments yield

$$\lim_{t \uparrow s} P(|Y_t - Y_s| > \varepsilon) = 0.$$

Thus Y is stochastically continuous. [34, Theorem 11.5] implies that there is a PII X with the desired properties.

□

APPENDIX C. BOCHNER INTEGRALS AND STOCHASTIC DIFFERENTIAL EQUATIONS IN FRÉCHET SPACES

Option price surfaces are interpreted as elements of the Fréchet space $L^1(\mathbb{R}_+, E)$ in Section 4. In order to derive existence and uniqueness results, we need to consider stochastic differential equations in such spaces, cf. Section C.2 below. These in turn rely on a properly defined Bochner integral., which is discussed in Section C.1.

C.1. **Bochner integration in Fréchet spaces.** Let F be a vector space and $(\|\cdot\|_n)_{n \in \mathbb{N}}$ an increasing sequence of separable semi-norms on F such that

(1) $\|x\|_n = 0\ \forall n \in \mathbb{N}$ holds only if $x = 0$,
(2) if $(x_k)_{k \in \mathbb{N}}$ is a $\|\cdot\|_n$-Cauchy sequence for all $n \in \mathbb{N}$, there exists $x \in F$ with $\lim_{k \to \infty} \|x_k - x\|_n = 0$ for any $n \in \mathbb{N}$.

Then

$$d(x,y) := \sum_{n \in \mathbb{N}} 2^{-n} (1 \wedge \|x - y\|_n)$$

defines a complete, translation-invariant, separable metric on the Fréchet space F.

Remark C.1. Let $x,y \in F$, $n \in \mathbb{N}$. Then

$$\begin{aligned} 1 \wedge \|x-y\|_n &\leq 2^n d(x,y) \quad \text{and} \\ d(x,y) &\leq \|x-y\|_n + 2^{-n}. \end{aligned}$$

Example C.2. We are mainly interested in the case that $F := L^1(\mathbb{R}_+, E)$ and $\|\cdot\|_n := \|\cdot\|_{n,n}$ as defined in Section 4 or, alternatively, $F = E$ itself in Lemma 4.1.

Fix a σ-finite measure space $(\Gamma, \mathscr{G}, \mu)$. The goal of this section is to define a Bochner integral $\int f d\mu$ for measurable functions $f : \Gamma \to F$ with values in the Fréchet space F, cf. Definition C.6 below. If f is *simple* and *integrable* in the sense that it is a linear combination of indicators of sets in $\mathscr{G}$ with finite μ-measure, the integral $\int f d\mu$ is naturally defined as a sum.

For fixed $n \in \mathbb{N}$ denote the set of measurable Bochner-integrable functions from $(\Gamma, \mathscr{G}, \mu)$ to the complete, separable, semi-normed space $(F, \|\cdot\|_n)$ by $\mathscr{L}^1((\Gamma, \mathscr{G}, \mu), (F, \|\cdot\|_n))$. Recall that a measurable function $f : \Gamma \to F$ is called *Bochner integrable* (relative to $\|\cdot\|_n$) if there is a sequence $(f^{(k)})_{k \in \mathbb{N}}$ of simple integrable functions with $\lim_{k\to\infty} \int \|f^{(k)} - f\|_n d\mu = 0$. Equivalently, $f : \Gamma \to F$ is measurable and $\int \|f\|_n d\mu < \infty$, cf. e.g. Lemma C.7 below. In this case there is some $x \in F$ such that

$$\lim_{k\to\infty} \left\| \int f^{(k)} d\mu - x \right\|_n = 0$$

for any such sequence. This element $\int f d\mu := x$ is called *($\|\cdot\|_n$-)Bochner integral* of f.

Note that we do not identify functions which are μ-a.e. identical. Therefore $\mathscr{L}^1((\Gamma, \mathscr{G}, \mu), (F, \|\cdot\|_n))$ together with the semi-norm $\|f\| := \int \|f\|_n d\mu$ is a complete semi-normed space, but in general not a Banach space. Moreover, *versions* of the $\|\cdot\|_n$-Bochner integral may differ by $\|\cdot\|_n$-distance zero.

The following lemma is needed in Section C.2.

Lemma C.3. *Let $\mathscr{O}_d, \mathscr{O}_n, n \in \mathbb{N}$ be the topologies generated by d and $\|\cdot\|_n$, respectively. Then $\mathscr{O}_n \subset \mathscr{O}_d$ for any $n \in \mathbb{N}$. For any $U \in \mathscr{O}_d$, $n_0 \in \mathbb{N}$ there is a sequence $(V_n)_{n \geq n_0}$ such that $V_n \in \mathscr{O}_n$ and $U = \cup_{n \geq n_0} V_n$. In particular, the Borel σ-field corresponding to the metric d is generated by the Borel σ-fields corresponding to the semi-norms $\|\cdot\|_n$, $n \geq n_0$.*

Proof. Let $n \in \mathbb{N}$ and $d_n(x,y) := 1 \wedge \|x-y\|_n$ for any $x,y \in F$. Observe that the semi-metric d_n generates the same topology as the semi-norm $\|\cdot\|_n$ because their balls of radius less or equal 1 coincide. Remark C.1 yields that $d_n \leq 2^n d$ and hence $\mathscr{O}_n \subset \mathscr{O}_d$.

The second inequality of Remark C.1 implies that for any $x \in F$, $\varepsilon > 0$ there is $n \geq n_0$ such that $B_n(x, \varepsilon/2) \subset B_d(x, \varepsilon)$, where $B_n(x, \varepsilon)$ resp. $B_d(x, \varepsilon)$ denote the balls centered at x with radius ε relative to the semi-norm $\|\cdot\|_n$ resp. the metric d. Let $U \in \mathscr{O}_d$. For any $x \in U$ choose $\varepsilon_x > 0$ such that $B_d(x, \varepsilon_x) \subset U$ and $n_x \in \mathbb{N}$ with $n_x \geq n_0$ such that $B_{n_x}(x, \varepsilon_x/2) \subset B_d(x, \varepsilon_x)$. For any $n \geq n_0$ define

$$V_n := \bigcup \{B_{n_x}(x, \varepsilon_x/2) : x \in U, n_x = n\} \in \mathscr{O}_n.$$

Since $V_n \subset U$, we have $\bigcup_{n \geq n_0} V_n \subset U$. Moreover,

$$U = \bigcup_{n \geq n_0} \{x : x \in U, n_x = n\} \subset \bigcup_{n \geq n_0} V_n.$$

Hence $U = \bigcup_{n \geq n_0} V_n$.

□

The $\|\cdot\|_n$-Bochner integrals are consistent in the following sense:

Lemma C.4. *Let $n \in \mathbb{N}$ and $f \in \mathscr{L}^1((\Gamma,\mathscr{G},\mu),(F,\|\cdot\|_n))$. Then $f \in \mathscr{L}^1((\Gamma,\mathscr{G},\mu),(F,\|\cdot\|_k))$ for $k = 1,\dots,n$. Moreover, if x is a version of the $\|\cdot\|_n$-Bochner integral $\int f d\mu$, then x is a version of the $\|\cdot\|_k$-Bochner integral $\int f d\mu$ for $k = 1,\dots,n$.*

Proof. Let $k \in \{1,\dots,n\}$. Then $\|\cdot\|_k \le \|\cdot\|_n$ and hence $\mathscr{O}_k \subset \mathscr{O}_n$. Thus f is measurable with respect to the Borel σ-field generated by $\|\cdot\|_k$. We have

$$\int \|f\|_k d\mu \le \int \|f\|_n d\mu < \infty$$

and hence $f \in \mathscr{L}^1((\Gamma,\mathscr{G},\mu),(F,\|\cdot\|_k))$. Let x be a version of the $\|\cdot\|_n$-Bochner integral $\int f d\mu$. Then there is a sequence $(f^{(k)})_{k\in\mathbb{N}}$ of simple integrable functions such that

$$\int \|f^{(k)} - f\|_n d\mu \to 0 \quad \text{and} \quad \left\| \int f^{(k)} d\mu - x \right\|_n \to 0$$

for $k \to \infty$. Since this also holds for k instead of n, we have that x is a version of the $\|\cdot\|_k$-Bochner integral $\int f d\mu$ as well.

□

We are now ready to define the desired integral for Fréchet space-valued functions.

Proposition C.5. *Let*

$$\begin{aligned} f \in \mathscr{L}^1((\Gamma,\mathscr{G},\mu),(F,d)) &:= \bigcap_{n\in\mathbb{N}} \mathscr{L}^1((\Gamma,\mathscr{G},\mu),(F,\|\cdot\|_n)) \\ &= \left\{ f : \Gamma \to F : f \text{ measurable with } \int \|f\|_n d\mu < \infty, n \in \mathbb{N} \right\}. \end{aligned}$$

Then there is one and only one $x \in F$ such that x is a version of the $\|\cdot\|_n$-Bochner integral $\int f d\mu$ for any $n \in \mathbb{N}$.

Proof. Since $f \in \mathscr{L}^1((\Gamma,\mathscr{G},\mu),(F,\|\cdot\|_n))$, there is a version x_n of the $\|\cdot\|_n$-Bochner integral $\int f d\mu$ for any $n \in \mathbb{N}$. Lemma C.4 yields that $d(x_n,x_m) \le \|x_n - x_m\|_k + 2^{-k} = 2^{-k}$ for any $n,m,k \in \mathbb{N}$ with $m,n \ge k$. Hence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (F,d). Since (F,d) is complete, it converges to some $x \in F$.

Let $n \in \mathbb{N}$. Lemma C.4 yieds that x_m is a version of the $\|\cdot\|_n$-Bochner integral $\int f d\mu$ and hence $\|x_m - x_n\|_n = 0$ for any $m \in \mathbb{N}$ with $m \ge n$. By Remark C.1 d-convergence implies $\|\cdot\|_n$-convergence. In particular, $\lim_{m\to\infty} \|x - x_m\|_n = 0$. Together we have

$$\|x - x_n\|_n = \lim_{m\to\infty} \|x_m - x_n\|_n = 0.$$

Hence x is a version of the $\|\cdot\|_n$-Bochner integral $\int f d\mu$.

Let $y \in F$ be a version of the $\|\cdot\|_n$-Bochner integral $\int f d\mu$ for any $n \in \mathbb{N}$. Then $\|x-y\|_n = 0$ for any $n \in \mathbb{N}$. Since the sequence of semi-norms is separating, we have $x = y$.

□

Definition C.6 (Bochner integral)**.** We call $\mathscr{L}^1((\Gamma,\mathscr{G},\mu),(F,d))$ the set of *Bochner-integrable* functions and $\int f d\mu := x$ from Proposition C.5 the corresponding *Bochner integral*.

The next lemma is due to Pettis and can be found in a slightly different version in e.g. in [16, Theorem II.2]. It is a characterisation of measurability which also holds in separable semi-metric spaces. However, we are mainly interested in the additional bound that can be imposed on the approximating sequence.

Lemma C.7. *Let $(E,\|\cdot\|)$ be a separable semi-normed space and $f : \Gamma \to E$. Then we have equivalence between:*

(1) *f is measurable,*
(2) *there is a sequence $(f^{(n)})_{n\in\mathbb{N}}$ of measurable functions such that* $\lim_{n\to\infty}\|f^{(n)}(t)-f(t)\|=0$ *for any $t\in\Gamma$,*
(3) *there is a sequence $(f^{(n)})_{n\in\mathbb{N}}$ of simple functions such that* $\lim_{n\to\infty}\|f^{(n)}(t)-f(t)\|=0$ *for any $t\in\Gamma$.*

In this case the sequence of simple functions can be chosen such that $\|f^{(n)}(t)\|\leq 2\|f(t)\|$ for any $t\in\Gamma$, $n\in\mathbb{N}$.

Proof. $1\Rightarrow 3$: Let f be measurable and $(x_n)_{n\in\mathbb{N}}$ a dense sequence in E with $x_0=0$, i.e. for any $y\in E$, $\varepsilon>0$ there is $n\in\mathbb{N}$ such that $\|x_n-y\|<\varepsilon$. Define closed sets

$$C_{n,k}:=\big\{y\in E:\|y-x_k\|=\min\{\|y-x_j\|:j=1,\dots,n\}\big\}$$

for any $n\in\mathbb{N}$ and $k=1,\dots,n$. Moreover, we define Borel-measurable sets $M_{n,k}:=C_{n,k}\setminus(C_{n,1}\cup\dots\cup C_{n,k-1})$ and simple functions

$$f^{(n)}:=\sum_{k=1}^{n}x_k 1_{f^{-1}(M_{n,k})}$$

for any $n\in\mathbb{N}$ and $k=1,\dots,n$. Then

$$\|f(t)-f^{(n)}(t)\|=\min\{\|f(t)-x_k\|:k=1,\dots,n\}\leq\|f(t)\|$$

and hence

$$\|f^{(n)}(t)\|\leq\|f^{(n)}(t)-f(t)\|+\|f(t)\|\leq 2\|f(t)\|$$

for any $n\in\mathbb{N}$, $t\in\Gamma$. Let $t\in\Gamma$ and $\varepsilon>0$. Then there is $n_0\in\mathbb{N}$ such that $\|x_{n_0}-f(t)\|<\varepsilon$. Hence

$$\|f^{(n)}(t)-f(t)\|\leq\|x_{n_0}-f(t)\|\leq\varepsilon,\quad n\geq n_0.$$

$3\Rightarrow 2$: This is obvious.

$2\Rightarrow 1$: We show that $f^{-1}(A)\in\mathscr{C}$ for any closed set $A\subset E$. We have $A=\bigcap_{n\in\mathbb{N}}A_n$ for the open sets

$$A_n:=\{x\in E:\exists y\in A:\|x-y\|<1/n\}.$$

Hence

$$f^{-1}(A)\subset\bigcap_{n\in\mathbb{N}}\bigcup_{N\in\mathbb{N}}\bigcap_{k\geq N}(f^{(k)})^{-1}(A_n)\subset\bigcap_{n\in\mathbb{N}}f^{-1}(A_n)\subset f^{-1}(\overline{A})=f^{-1}(A),$$

which implies

$$f^{-1}(A)=\bigcap_{n\in\mathbb{N}}\bigcup_{N\in\mathbb{N}}\bigcap_{k\geq N}(f^{(k)})^{-1}(A_n)\in\mathscr{C}.$$

□

C.2. **Stochastic differential equations with Fréchet space-valued processes.** Let (F,d) denote the Fréchet space of the previous section. We identify right-continuous, increasing functions $X:\mathbb{R}_+\to\mathbb{R}$ with their corresponding Lebesgue-Stieltjes measure μ on $\mathbb{R}_+$. For $t\in\mathbb{R}_+$ and $\varphi:\mathbb{R}_+\to F$ we write

$$\varphi\in\mathscr{L}^1\big(([0,t],X),(F,d)\big)$$

if $\varphi 1_{[0,t]}\in\mathscr{L}^1\big((\mathbb{R}_+,\mathscr{B}(\mathbb{R}_+),\mu)),(F,d)\big)$. Moreover, we write

$$\int_0^t\varphi_s dX_s:=\int\varphi 1_{[0,t]}d\mu\in F$$

for the integral from Definition C.6.

For right-continuous increasing processes X and measurable functions $\varphi : \Omega \times \mathbb{R}_+ \to \mathbb{R}$, both $\varphi \in \mathscr{L}^1(([0,t],X),(F,d))$ and $\int_0^t \varphi_s dX_s$ are to be interpreted in a pathwise sense, i.e. for any fixed $\omega \in \Omega$.

Lemma C.8. *Let $X : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing right-continuous process. Moreover, let $\varphi : (\Omega \times \mathbb{R}_+, \mathscr{A}) \to (F,d)$ be Borel measurable with $\varphi \in \mathscr{L}^1(([0,T],X),(F,d))$ for any $T \in \mathbb{R}_+$, where $\mathscr{A}$ denotes the optional σ-field. Then $Y_t := \int_0^t \varphi_s dX_s$ defines an adapted càdlàg process Y.*

Proof. The dominated convergence theorem applied to the $\|\cdot\|_n$-Bochner integrals yields the càdlàg property of the process Y.

We now show that Y is adapted. Let $T \in \mathbb{R}_+$ and $n \in \mathbb{N}$. Lemma C.3 yields that $\Omega \times [0,T] \to F$, $(\omega,t) \mapsto \varphi_t(\omega)$ is $\mathscr{F}_T \otimes \mathscr{B}([0,T])$-$\mathscr{B}_n$-measurable, where $\mathscr{B}_n$ denotes the Borel σ-field generated by $\|\cdot\|_n$. Lemma C.7 yields that there is a sequence of simple $\mathscr{F}_T \otimes \mathscr{B}([0,T])$-$\mathscr{B}_n$-measurable functions $(\varphi^{(k)})_{k\in\mathbb{N}}$ such that

$$\begin{aligned} \|\varphi_t^{(j)}(\omega)\|_n &\leq 2\|\varphi_t(\omega)\|_n, \\ \|\varphi_t^{(k)}(\omega) - \varphi_t(\omega)\|_n &\to 0 \quad \text{for } k \to \infty \end{aligned}$$

for any $j \in \mathbb{N}$, $\omega \in \Omega$, $t \in [0,T]$. The random variable $\int_0^t \varphi_s^{(k)} dX_s$ is $\mathscr{F}_T$-$\mathscr{B}_n$-measurable for any $k \in \mathbb{N}$. Proposition C.5 yields that the F-valued Bochner integral $Z := \int_0^t \varphi_s dX_s$ is a version of the $\|\cdot\|_n$-Bochner integral. The integral inequality for Bochner integrals and the dominated convergence theorem yield

$$\lim_{k\to\infty} \left\| \left(\int_0^t \varphi_s^{(k)} dX_s \right)(\omega) - Z(\omega) \right\|_n \leq \lim_{k\to\infty} \left(\int_0^t \|\varphi_s^{(k)} - \varphi_s\|_n dX_s \right)(\omega) = 0$$

for any $\omega \in \Omega$. Hence Lemma C.7 implies that Z is $\mathscr{F}_T$-$\mathscr{B}_n$-measurable. Since n was chosen arbitraily, Lemma C.3 yields that Z is $\mathscr{F}_T$-$\mathscr{B}_d$-measurable, where $\mathscr{B}_d$ denotes the Borel σ-field generated by d.

□

We now turn to existence and uniqueness of solutions to Banach space-valued SDE's. As usual, this follows e.g. under Lipschitz conditions.

Theorem C.9. *Let $X = (X^1, \ldots, X^d)$ be an $\mathbb{R}^d$-valued right-continuous process whose components are nonnegative and increasing. Moreover, let $(E, \|\cdot\|)$ be a separable Banach space, $x \in E$ and $a : \mathbb{R}_+ \times E \to E^d$ measurable such that*

$$\sup_{s\in[0,T]} \sum_{i=1}^d \|a_i(s,0)\| < \infty$$

and

$$\sup_{s\in[0,T]} \sum_{i=1}^d \|a_i(s,y) - a_i(s,z)\| < C_T \|y-z\|$$

for any $T > 0$, some $C_T > 0$ and any $y,z \in E$. Then there is a unique E-valued càdlàg process Y such that

$$Y_t = x + \sum_{i=1}^d \int_0^t a_i(s, Y_{s-}) dX_s^i, \quad t \geq 0,$$

where the right-hand side contains pathwise Bochner integrals. This process is adapted to the filtration generated by X.

Proof. The proof uses the standard Picard iteration scheme (cf. [21, Theorem I.1.1]) to construct an adapted solution and Grönwall's inequality (cf. [17, Theorem A.5.1]) to show uniqueness among all possible solutions. For ease of notation we assume that $d = 1$. The general case follows along the same lines.

Let $(\mathscr{F}_t)_{t\geq 0}$ be the right-continuous filtration generated by X. W.l.o.g. we may assume that $X_0 = 0$. Let $g : \mathbb{R}_+ \to E$ be càdlàg. For any $T \in \mathbb{R}_+$ and $s \leq T$ we have

$$\begin{aligned}\|a(s,g(s-))\| &\leq C_T\|g(s-)\| + \|a(s,0)\| \\ &\leq \sup_{s\in[0,T]} (C_T\|g(s)\| + \|a(s,0)\|) \\ &< \infty.\end{aligned}$$

Hence the measurable function $f : \mathbb{R}_+ \to E$, $s \mapsto a(s,g(s-))$ is bounded on any $[0,T]$. Thus f is integrable on compact sets with respect to any finite Borel measure, e.g. with respect to the Lebesgue-Stieltjes measure of X. In particular, if V is an adapted E-valued càdlàg process, then the pathwise integrals

$$\begin{aligned}\Phi(V)_t &:= x + \int_0^t a(s,V_{s-})dX_s, \\ \Phi^\tau(V)_t &:= x + \int_0^t 1_{[\![0,\tau]\!]}(s)a(s,V_{s-})dX_s, \\ \Phi^{\tau-}(V)_t &:= x + \int_0^t 1_{[\![0,\tau[\![}(s)a(s,V_{s-})dX_s\end{aligned}$$

are càdlàg adapted process for any stopping time τ, cf. Lemma C.8. Observe that if V is a fixed point of $\Phi^{\tau-}$, then $W := \Phi^\tau(V)$ is a fixed point of Φ^τ.

Let $T \in \mathbb{N}$. [25, I.1.18 and I.1.28] yield that

$$\tau_n := T \wedge \inf\left\{t \geq 0 : X_t \geq \frac{n}{2C_T}\right\}$$

is a stopping time for any $n \in \mathbb{N}$. Moreover, $\tau_0 = 0$, $\tau_n \leq \tau_{n+1}$, $\tau_n = T$ for large n and we have

$$1_{\{\tau_n \neq T\}} C_T |X_{\tau_{n+1}-} - X_{\tau_n}| \leq \frac{1}{2}$$

for any $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ such that there is a fixed point V^n of Φ^{τ_n}. For any E-valued process W define the random variable $\|W\|_\infty := \sup_{s\leq T}\|W_s\|$ and let $\mathscr{V}$ be the set of adapted càdlàg processes which coincide with V^n until τ_n. Observe that $\Phi^{\tau_{n+1}}(\mathscr{V}) \subset \mathscr{V}$ because it maps adapted càdlàg processes to adapted càdlàg processes and V^n is a fixed point of Φ^{τ_n}. For $W^1, W^2 \in \mathscr{V}$ we have

$$\begin{aligned}\|\Phi^{\tau_{n+1}-}(W^1)_t - \Phi^{\tau_{n+1}-}(W^2)_t\| &\leq \int_0^t 1_{[\![0,\tau_{n+1}[\![}(s)\|a(s,W^1_{s-}) - a(s,W^2_{s-})\|dX_s \\ &\leq \int_0^t 1_{[\![\tau_n,\tau_{n+1}[\![}(s)\|a(s,W^1_{s-}) - a(s,W^2_{s-})\|dX_s \\ &\leq 1_{\{\tau_n\neq T\}} C_T \|W^1 - W^2\|_\infty |X_{\tau_{n+1}-} - X_{\tau_n}| \\ &< \frac{1}{2}\|W^1 - W^2\|_\infty\end{aligned}$$

for any $t \in [0,T]$ and thus we obtain

$$\|\Phi^{\tau_{n+1}-}(W^1) - \Phi^{\tau_{n+1}-}(W^2)\|_\infty \leq \frac{1}{2}\|W^1 - W^2\|_\infty.$$

Thus $\Phi^{\tau_{n+1}-}$ is a contraction on the set $\mathscr{V}$, which implies that there is a fixed point $W \in \mathscr{V}$ of $\Phi^{\tau_{n+1}-}$. In particular, $\Phi^{\tau_{n+1}}$ has an adapted càdlàg fixed point as well, namely $V^{n+1} := \Phi^{\tau_{n+1}}(W)$. Define the adapted càdlàg process $U_t^T := \lim_{n\to\infty} V_t^n$. This process U^T is a fixed point of Φ^T. Its pointwise limit $Y_t := \lim_{T\to\infty} U_t^T$ is an adapted càdlàg process and a fixed point of Φ.

Let Z be another pathwise solution, i.e. Z is E-valued, càdlàg, and $Z_t = x + \int_0^t a(s, Z_{s-})dX_s$, $t \geq 0$. Fix $\omega \in \Omega$ and define

$$f(t) := \sup_{s\in[0,t)} \|Y_s(\omega) - Z_s(\omega)\|$$

for all $t \in \mathbb{R}_+$. f is finite because Y and Z are càdlàg. Moreover, we have

$$\begin{aligned} f(t) &= \sup_{s\in[0,t)} \|\Phi(Y)_s(\omega) - \Phi(Z)_s(\omega)\| \\ &\leq C_T \left(\int_0^t f(s)dX_s \right)(\omega) \end{aligned}$$

for $t \leq T$. [17, Theorem A.5.1] yields $f(t) = 0$ for all $t \geq 0$. Thus $Z = Y$.

□

If a growth condition holds, the Lipschitz condition can be relaxed as usual to a local version.

Corollary C.10. *Let $X = (X^1, \ldots, X^d)$ be an $\mathbb{R}^d$-valued right-continuous process whose components are nonnegative and increasing. Moreover, let $(E, \|\cdot\|)$ be a separable Banach space, $x \in E$ and $a : \mathbb{R}_+ \times E \to E^d$ measurable such that for any $T \in \mathbb{R}_+$ there is $C_T < \infty$ such that*

$$\sup_{s\in[0,T]} \sum_{i=1}^d \|a_i(s,y)\| \leq C_T(1 + \|y\|), \quad y \in E$$

and for any $K, T \in \mathbb{R}_+$ some $C_{K,T} < \infty$ such that

$$\sup_{s\in[0,T]} \sum_{i=1}^d \|a_i(s,y) - a_i(s,z)\| < C_{K,T}\|y - z\|$$

for any $y, z \in E$ with $\|y\| \leq K$, $\|z\| \leq K$. Then there is a unique E-valued càdlàg process Y such that

$$Y_t = x + \sum_{i=1}^d \int_0^t a_i(s, Y_{s-})dX_s^i, \quad t \geq 0. \tag{C.1}$$

This process is adapted to the filtration generated by X.

Proof. Again we consider $d = 1$ for the proof. Fix $K \in \mathbb{R}_+$. Choose $a^K : \mathbb{R}_+ \times E \to E$ such that $a^K(t,y) = a(t,y)$ for $\|y\| \leq K$ and such that the Lipschitz condition of Theorem C.9 holds for a^K, e.g.

$$a_K(t,y) := a\left(t, y\frac{1 + (\|y\| \wedge K)}{1 + \|y\|}\right).$$

Denote the corresponding solution to the SDE by Y^K. Grönwall's inequality yields as in the proof of Theorem C.9 that for $K, L \in \mathbb{R}_+$ the solutions Y^K and Y^L coincide till the norm of either of the two processes exceeds $K \wedge L$. Hence, setting $Y_t = Y_t^K$ for $t \in \mathbb{R}_+$ with $\sup_{s\in[0,t]} \|Y_s^K\| \leq K$ yields a well-defined adapted càdlàg process, which may however explode at a finite time. However, the linear growth condition and Grönwall's inequality yield that $\sup_{t\in[0,T]} \|Y_t^K\| \leq c_T$ for some finite random variable c_T which does not depend on

K. Consequently, Y is defined on $\mathbb{R}_+$ and it solves SDE (C.1). Uniqueness follows as in the proof of Theorem C.9.

□

Corollary C.10 can be extended to Fréchet-valued processes and their respective integrals.

Corollary C.11. *Let $X = (X^1, \ldots, X^d)$ be an $\mathbb{R}^d$-valued right-continuous process whose components are nonnegative and increasing. Moreover, let $x \in F$, and $a : \mathbb{R}_+ \times F \to F^d$ measurable such that for any $T \in \mathbb{R}_+$ there is $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ there is $C < \infty$ with*

$$\sup_{s\in[0,T]} \sum_{i=1}^{d} \|a_i(s,y)\|_n \leq C(1+\|y\|_n), \quad y \in F$$

and for any $n \geq n_0$, $K \in \mathbb{R}_+$ there is some $C_K < \infty$ with

$$\sup_{s\in[0,T]} \sum_{i=1}^{d} \|a_i(s,y) - a_i(s,z)\|_n < C_K \|y-z\|_n$$

for $y,z \in F$ with $\|y\|_n \leq K$, $\|z\|_n \leq K$. Then there is a unique F-valued càdlàg process Y such that

$$Y_t = x + \sum_{i=1}^{d} \int_0^t a_i(s, Y_{s-}) dX_s^i, \quad t \geq 0,$$

where the right-hand side refers to the pathwise Bochner integral introduced above. This process is adapted to the filtration generated by X.

Proof. As before, we consider $d = 1$ for ease of notation.

Existence for a modified equation. Fix $T \in \mathbb{R}_+$ and define

$$\widetilde{a} : \mathbb{R}_+ \times F \to F^d, \quad (t,x) \mapsto 1_{[0,T]}(t) a(t,x).$$

By assumption there is an $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, $K \geq 0$ there are $C, C_K < \infty$ such that

$$\sup_{s\in\mathbb{R}_+} \|\widetilde{a}(s,y)\|_n \leq C(1+\|y\|_n), \quad y \in F, \tag{C.2}$$

$$\sup_{s\in\mathbb{R}_+} \|\widetilde{a}(s,x) - \widetilde{a}(s,z)\|_n < C_K \|x-z\|_n \tag{C.3}$$

for any $x,z \in F$ with $\|x\|_n \leq K$, $\|z\|_n \leq K$. Let $n \geq n_0$, $K \in \mathbb{R}_+$ and choose $C, C_K > 0$ such that (C.2, C.3) hold. Consider the factor space $G_n := F/N_n$ with $N_n := \{y \in F : \|y\|_n = 0\}$ and the corresponding factor norm $\|\cdot\|_{c,n}$. Then $(G_n, \|\cdot\|_{c,n})$ is a separable Banach space. Let $\rho_n : F \to G_n$ be the factor mapping. Observe that

$$\|\rho_n(\widetilde{a}(s,x)) - \rho_n(\widetilde{a}(s,y))\|_{c,n} = \|\widetilde{a}(s,x) - \widetilde{a}(s,y)\|_n \leq C_K \|x-y\|_n = 0$$

for any $s \leq T$, $K \in \mathbb{N}$, $x,y \in F$ with $\|x\|_n \leq K$, $\|y\|_n \leq K$, and $\|x-y\|_n = 0$. Thus the function

$$b^{(n)} : \mathbb{R}_+ \times G_n \to G_n, \quad (s,[x]) \mapsto \rho_n(\widetilde{a}(s,x))$$

is well defined. Moreover, $b^{(n)}$ is measurable,

$$\sup_{s\in\mathbb{R}_+} \|b^{(n)}(s,y)\|_{c,n} < C(1+\|y\|_n), \quad y \in G_n$$

and

$$\sup_{s\in\mathbb{R}_+} \|b^{(n)}(s,y) - b^{(n)}(s,z)\|_{c,n} < C_K \|y-z\|_{c,n}$$

for any $y,z \in G_n$ with $\|x\|_n \le K$, $\|y\|_n \le K$. Then Corollary C.10 yields that there is a unique adapted càdlàg solution $\hat{Y}$ to the SDE

$$\hat{Y}_t = \rho_n(x) + \int_0^t b^{(n)}(s, \hat{Y}_{s-})dX_s.$$

Let $\varphi_n : G_n \to F$ be a linear choice of representants. Then φ_n is isometric and hence $Y^{(n)} := \varphi_n(\hat{Y})$ is a version of $x + \int_0^t \widetilde{a}(s, Y_{s-}^{(n)})dX_s$, where we refer to the $\|\cdot\|_n$-Bochner integral. Lemma C.4 yields that $Y_t^{(n)}$ is a version of $x + \int_0^t \widetilde{a}(s, Y_{s-}^{(n)})dX_s$, where the integral can be interpreted as a $\|\cdot\|_k$-Bochner integral for $k = 1, \ldots, n$. Thus $Z := \rho_k(Y^{(n)})$ is the unique solution to the SDE

$$Z_t = \rho_k(x) + \int_0^t b^{(k)}(s, Z_{s-})dX_s$$

for $k = n_0, \ldots, n$, $t \in \mathbb{R}_+$. Hence $\|Y_t^{(n)} - Y_t^{(k)}\|_k = 0$ for $k = n_0, \ldots, n$, $t \in \mathbb{R}_+$. Thus $(Y_t^{(n)}(\omega))_{n \ge n_0}$ is a $\|\cdot\|_k$-Cauchy sequence for any $k \ge n_0$ and any $\omega \in \Omega$. Hence it is a d-Cauchy sequence and we define

$$Y_t(\omega) := \lim_{n\to\infty} Y_t^{(n)}(\omega)$$

relative to the metric d and any $\omega \in \Omega$, $t \in \mathbb{R}_+$. Since $\|Y_t^{(k)} - Y_t\|_k = 0$, we have

$$Y_t = x + \int_0^t \widetilde{a}(s, Y_{s-})dX_s \tag{C.4}$$

for some version of the $\|\cdot\|_k$-Bochner integral. Since this is true for any $k \ge n_0$, we conclude that the equation holds also relative to the Fréchet space-valued Bochner integral. The càdlàg property of Y follows from the càdlàg properties of $Y^{(n)}$ relative to the semi-norm $\|\cdot\|_n$ for any $n \ge n_0$. Random variable Y_t is $\mathscr{F}_t$-$\mathscr{B}_n$-measurable for any $t \in \mathbb{R}_+$, $n \ge n_0$ because this holds for $\hat{Y}_t = \rho_n(Y_t)$ by Corollary C.10. Lemma C.3 yields that Y is adapted.

Uniqueness for the modified equation. Let Z be an arbitrary solution to (C.4) in the sense of (F,d)-space valued integrals. Let $n \ge n_0$ and define the factor space G_n, the factor mapping ρ_n, and the function $b^{(n)}$ as above. Then we have

$$\rho_n(Z_t) = \rho_n(x) + \int_0^t b^{(n)}(s, \rho_n(Z_{s-}))dX_s$$

for any $t \in \mathbb{R}_+$. Corollary C.10 yields that $\rho_n \circ Z = \rho_n \circ Y$ or, equivalently, $\|Z_t - Y_t\|_n = 0$ for any $t \in \mathbb{R}_+$. Since this is true for any $n \ge n_0$, we have $Z = Y$.

Consistency of solutions: Let $0 \le T_1 \le T_2 < \infty$ and

$$\widetilde{a}_1 : \mathbb{R}_+ \times F \to F^d, \quad (t,x) \mapsto 1_{[0,T_1]}(t)a(t,x),$$
$$\widetilde{a}_2 : \mathbb{R}_+ \times F \to F^d, \quad (t,x) \mapsto 1_{[0,T_2]}(t)a(t,x).$$

Let $Y^{(i)}$ be the unique (F,d)-valued solution to the SDE

$$Y_t^{(i)} = x + \int_0^t \widetilde{a}_i(s, Y_{s-}^{(i)})dX_s, \quad t \in \mathbb{R}_+ \tag{C.5}$$

for $i = 1,2$, which has been constructed above. Then the stopped process $(\widetilde{Y}^{(2)})^{T_1}$ solves the (F,d)-valued SDE (C.5) for $i = 1$, which implies $\widetilde{Y}^{(1)} = (\widetilde{Y}^{(2)})^{T_1}$.

Existence and uniqueness: For any $T \in \mathbb{N}$ let $Y^{(T)}$ be the unique (F,d)-valued solution to the SDE

$$Y_t^{(T)} = x + \int_0^t 1_{[0,T]}(t)a(s, Y_{s-}^{(T)})dX_s, \quad t \in \mathbb{R}_+. \tag{C.6}$$

Define $Y_t := Y_t^{(T)}$ for any $T \in \mathbb{N}$, $t \leq T$. Process Y is well defined due to consistency of solutions. Observe that Y is a solution to the (F,d)-valued SDE

$$Y_t = x + \int_0^t a(s, Y_{s-}) dX_s, \quad t \in \mathbb{R}_+.$$

If Z is any solution to this (F,d)-valued SDE, then Z^T solves (C.6) for any $T \in \mathbb{N}$. This yields $Z_t = Y_t^{(T)} = Y_t$, $t \in [0,T]$.

□

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